Generating Einstein's Solutions via Hypergeometric Equation

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Abstract

Exact solutions to Einstein field equations in spherically symmetric gravitational fields are obtained for an anisotropic matter with specified forms for the anisotropic factor and one of the gravitational potentials. The solution of the Einstein field equations is reduced to a difference equation with variable rational coefficients which can be solved in general. It is possible to obtain general class of solutions in terms of special functions and elementary functions for different partial geometries.

Keywords: Einstein field equations, exact solutions, anisotropic matter

Introduction

Exact solutions of the Einstein field equations for an anisotropic matter are important in the description of relativistic astrophysical processes. In recent years a number of authors have studied exact solutions to the Einstein field equations corresponding to the anisotropic matter where the radial component of the pressure differs from the angular component. The gravitational field is taken to be spherically symmetric and static since these solutions may be applied to relativistic stars. A number of researchers have examined how anisotropic matter affects critical mass, critical surface redshift and stability of highly compact bodies. These investigations are contained in the paper by Dev and Gleiser (2003). Some researchers have suggested that anisotropy may be important in understanding the gravitational behavior of boson stars and the role of strange matter with densities higher than neutron stars. Mark and Harko (2002) and Sharma and Mukherjee (2002) suggest that anisotropy is crucial ingredient in the description of dense stars with strange matter

In order to solve the field equations, various restrictions have been placed by investigators on the geometry of spacetime and the matter content. Mainly two distinct procedures have been adopted to solve these equations for spherically symmetric static manifolds. Firstly, the coupled differential equations are solved by computations after choosing an equation of state. Secondly, the exact Einstein solutions can be obtained by specifying the geometry. We follow the later technique in an attempt to find solutions in terms of special functions and elementary functions that are suitable for description of relativistic stars. This approach was recently used by Chaisi and Maharaj (2005) that yield a solution in terms of elementary functions. This solution have considered by many authors in the analysis of gravitational behavior of compact objects, and the study of anisotropy under strong gravitational fields. Hence the approach followed in this paper has proved to be a fruitful avenue for generating new exact solution for describing the interior spacetimes of relativistic spheres.

The objective of this paper is to provide systematically a rich family of Einstein field equations with anisotropic matter which satisfy the physical properties similar to the recent treatment of Maharaj and Komathiraj (2007). In Section 2, the Einstein field equations for the static spherically symmetric line element with anisotropic matter is expressed as an equivalent set of differential equations utilizing a transformation from Durgapal and Bannerji (1983). We chose particular forms for one of the gravitational potentials and the anisotropic factor, which enables us to obtain the condition of pressure anisotropy in the remaining gravitational potential in Section 3. We assume a solution in a series form which yields recurrence relation, which we manage to solve from first principle. It is then possible to exhibit exact solutions to the Einstein field equations. We demonstrate that the exact solutions to the Einstein field equations in terms of hyper geometric functions are possible and we generate two linearly independent classes of solutions by determining the specific restriction on the parameters in section 4. Finally in section 5, we discuss the physical feature of the solutions.

1. Field equations

Assume that the spacetime is spherically symmetric and static which is consistent with the study of anisotropic compact objects. Therefore there exists coordinates (t, r, θ, φ) such that the line element is of the form

$$ds^2 = -e^{2f(r)}dt^2 + e^{2g(r)}dr^2 + r^2(d\theta^2 + \sin\theta\, d\phi^2) \tag{1}$$

In Schwarzschild coordinates (t, r, θ, φ), where $f(r)$ and $g(r)$ are arbitrary functions. For a perfect fluid the Einstein field equations can be written in the form

$$\frac{1}{r^2}(1-e^{-2g})+\frac{2g'}{r}e^{-2g}=\mu \qquad (2)$$

$$-\frac{1}{r^2}(1-e^{-2g})+\frac{2f'}{r}e^{-2g}=p_r \qquad (3)$$

$$e^{-2g}\left(f''+f'^2+\frac{f'}{r}-f'g'-\frac{g'}{r}\right)=p_t \qquad (4)$$

where primes denote differentiation with respect to r. In equation (2) - (4), the quantity μ is the energy density, p_r is the radial pressure and p_t is the tangential pressure. The Einstein field equations (2) - (4) describe the gravitational behavior for an anisotropic imperfect fluid. For matter distributions with $p_r = p_t$ (isotropic pressures), the Einstein's equations for a perfect fluid may be regained from (2) - (4). A different but equivalent form of the field equations is generated by introducing a new independent variable x and two new functions y and Z.

These are given by

$$A^2y^2(x)=e^{2f},\ Z(x)=e^{-2g},\ x=Cr^2 \qquad (5)$$

In equation (5), A and C are arbitrary constants. Under the transformation (5), the system (2) – (4) becomes

$$\frac{1-Z}{x}-2\dot{Z}=\frac{\mu}{C} \qquad (6)$$

$$4Z\frac{\dot{y}}{y}=\frac{Z-1}{x}=\frac{p_r}{C} \qquad (7)$$

$$4Zx^2\ddot{y}+2x^2\dot{Z}\dot{y}+\left(\dot{Z}x-Z+1-\frac{\Delta x}{C}\right)y=0 \qquad (8)$$

$$\Delta=p_t-p_r \qquad (9)$$

where dots denote differentiations with respect to x. The quantity Δ is defined as the measure of anisotropy or anisotropy factor. The Einstein field equations as expressed in (6) - (9) is a system of four nonlinear equations in the six unknown $(\mu, p_r, p_t, Z, y, \Delta)$. The advantage of this system lies in the fact that a solution can, upon a suitable substitution of Z and Δ, be readily obtain by integrating (8) which is second order and linear in y.

2. Master equation

We solve the Einstein field equations (6) – (9) by making explicit choices for the gravitational potential Z and the measure of anisotropy Δ. For the metric function Z we make the choice

$$Z=\frac{1+kx}{1+x} \qquad (10)$$

The potential Z in (10) is regular at the origin and continuous in the stellar interior of the star for a wide range of value of the parameter k. Therefore the form chosen in (10) are physically acceptable. This specific choice for Z simplifies the integration process. Substitution of (10) into (8) leads to the equation

$$4(1+kx)(1+x)\ddot{y}-2(1-k)\dot{y}+[(1-k)-\frac{\Delta}{x}(1+x)^2]y=0 \qquad (11)$$

It is necessary to specify the anisotropic factor Δ to solve (11). A variety of choices for Δ is possible but only a few are physically reasonable which generate closed form solutions. The differential equation (11) can be reduced to simpler form if we let

$$\Delta = \frac{ax}{(1+x)^2} \quad (12)$$ where a is a real constant. Upon substituting the choice (12) into equation (11) we obtain

$$4(1+kx)(1+x)\ddot{y} - 2(1-k)\dot{y} + [(1-k)-a]y = 0 \quad (13)$$

It is convenient to introduce the new variable $z = (1 + x)$ in (13) to yield

$$4z(1-k+kz)\frac{d^2Y}{dz^2} - 2(1-k)\frac{dY}{dz} + [(1-k)-a]Y = 0 \quad (14)$$

where Y is a function of z.

The differential equation (14) is the master equation of the system (6) – (9). Two categories of solutions are possible for $(k = 1)$ and $(k \neq 1)$.

Case I: $\boldsymbol{k = 1}$

In this case (13) becomes the Euler-Cauchy equation with solution

$$y = c_1(1+x)^{\frac{1+\sqrt{1+a}}{2}} + c_2(1+x)^{\frac{1-\sqrt{1+a}}{2}}$$

In terms of the original variable x, where c_1 and c_2 are two arbitrary constants.

Case II: $\boldsymbol{k \neq 1}$

With $k \neq 1$ the master equation (14) can be solved using the method of Frobenius. As the point $z = 0$ is a regular singular point of (14), there exist two linearly independent solutions of the form of a power series with centre $z = 0$. These solutions can be generated using the method of Frobenius. Therefore we can assume

$$Y = \sum_{i=0}^{\infty} a_i z^{i+b}, \; a_0 \neq 0 \quad (15)$$

In equation (15) a_i are the coefficients of the series and b is a constant. For a legitimate solution the coefficients a_i and the parameter b should be determined explicitly. On substituting (15) into (14), we obtain

$$2a_0b(1-k)(2b-3)z^{b-1}\sum_{i=0}^{\infty}\{2(1-k)(i+b-1)(2i+2b-1)a_{i+1}$$
$$+[4k(i+b)(i+b-1)+(1-k-a)]a_i\}z^{b+i} = 0$$

The coefficients of the various powers of z must vanish. Equating the coefficient of z^{b-1} to zero we obtain $2a_0b(1-k)(2b-3) = 0$. Since $a_0 \neq 0$, $b = 0$ or $b = 3/2$. Equating the coefficient of z^{b+i} to zero, we obtain

$$a_{i+1} = \frac{4k(i+b)(i+b-1)+(1-k-a)}{2(k-1)(i+b+1)(2i+2b-1)}a_i, \; i \geq 0$$

which is the recurrence formula, or difference equation, governing the structure of the solution. It is possible to express the general coefficient a_i in terms of the leading coefficient a_0 by establishing a general structure for the coefficient by considering the leading terms. These coefficients generate the pattern

$$a_{i+1} = \prod_{p=0}^{i} \frac{4k(p+b)(p+b-1)+(1-k-a)}{2(k-1)(p+b+1)(2p+2b-1)} a_0 \tag{16}$$

Now it is possible to generate two linearly independent solutions to (14) with the assistance of (15) and (16). For the parameter value b=0, the first solution is given by

$$Y_1 = a_0\left[1+\sum_{i=0}^{\infty} \prod_{p=0}^{i} \frac{4kp(p-1)+(1-k-a)}{2(k-1)(p+1)(2p-1)} z^{i+1}\right]$$

For the parameter value $b = 3/2$, the second solution has the form

$$Y_2 = a_0 z^{\frac{3}{2}}\left[1+\sum_{i=0}^{\infty} \prod_{p=0}^{i} \frac{k(2p+3)(2p+1)+(1-k-a)}{(k-1)(2p+5)(2p+2)} z^{i+1}\right]$$

Thus the general solution to the differential equation (14), for the choices in (10) and (12) is given as

$$Y = d_1Y_1 + d_2Y_2$$

where d_1 and d_2 are constants. In terms of the original variable x, the function Y given above becomes

$$y = A\left[1+\sum_{i=0}^{\infty}\prod_{p=0}^{i} \frac{4kp(p-1)+(1-k-a)}{2(k-1)(p+1)(2p-1)}(1+x)^{i+1}\right]$$

$$+B(1+x)^{\frac{3}{2}}\left[\begin{array}{l}1+\sum_{i=0}^{\infty}\prod_{p=0}^{i} \frac{k(2p+3)(2p+1)+(1-k-a)}{(k-1)(2p+5)(2p+2)} \\ (1+x)^{i+1}\end{array}\right] \tag{17}$$

Where $A = d_1a_0$, $B = d_2a_0$ Thus we have found the general series solution (17) to the differential equation (11). This solution is expressed in terms of a series with real arguments unlike the complex arguments given by software packages

3. Solution in terms of elementary functions

The general solution (18) is given in the form of a series which define special functions. It is possible for the general solution to be written in terms of elementary functions in closed form which is a more desirable form for the physical description of a relativistic star. If we introduce the transformation in (14), we obtain

$$1+x = KX,\ K = \frac{k-1}{k},\ Y(X) = y(x)$$

$$4X(1-X)\frac{d^2Y}{dX^2} - 2\frac{dY}{dX} + (K+\hat{a})Y = 0,\ \hat{a} = \frac{a}{k} \tag{18}$$

which is a special case of hypergeometric differential equation. It is possible to obtain two linearly independent solutions to (18) in terms of hypergeometric functions $Y_1$and Y_2. These two functions are given by

$$Y_1 = F\left[-\frac{1}{2}-\frac{1}{2}\sqrt{1+K+\hat{a}}, -\frac{1}{2}+\frac{1}{2}\sqrt{1+K+\hat{a}}, -\frac{1}{2}, X\right] \text{and}$$

$$Y_2 = X^{\frac{3}{2}}F\left[1-\frac{1}{2}\sqrt{1+K+\hat{a}}, 1+\frac{1}{2}\sqrt{1+K+\hat{a}}, \frac{5}{2}, X\right]$$

It is well known that hypergeometric functions can be written in terms of elementary functions for particular parameter values. This statement is also true for these two hypergeometric functions. Consequently two sets of general solutions in terms of elementary functions can be found by restricting the range of values of K and $\tilde{\alpha}$ .Thus we can express the first category of solution to (14) as

$$y = A\left(\frac{K-1-x}{K}\right)^{\frac{1}{2}}\left[\begin{array}{l} 4(n+1)\sum_{i=1}^{n+1}\frac{(-4)^{i-1}(2i-1)(n+i)!}{(2i)!(n-i+1)!} \\ \times\left(\frac{1+x}{K}\right)^{i} + 1 \end{array}\right] + B\left(\frac{1+x}{K}\right)^{\frac{3}{2}}\left[\begin{array}{l} \frac{3}{n+1}\sum_{i=1}^{n}\frac{3(-4)^{i}(2i+2)(n+i+1)!}{(2i+3)!(n-i)!} \\ \times\left(\frac{1+x}{K}\right)^{i} + 1 \end{array}\right] \qquad (19)$$

For $K + \alpha = (2n+3)(2n+1)$.

The second category of solution is given by

$$y = A\left(\frac{K-1-x}{K}\right)^{\frac{1}{2}}\left[\begin{array}{l} \frac{3}{n(n-1)}\sum_{i=1}^{n-2}\frac{(-4)^{i}(2i+2)(n+i)!}{(2i+3)!(n-i-2)!} \\ \times\left(\frac{1+x}{K}\right)^{i} + 1 \end{array}\right] + B\left[4n(n-1)\sum_{i=1}^{n}\frac{3(-4)^{i-1}(2i-1)(n+i-2)!}{(2i)!(n-i)!}\left(\frac{1+x}{K}\right)^{i} + 1\right] \qquad (20)$$

For $K + \alpha = 4n(n-1)$.

Therefore two categories of solutions in terms of elementary functions can be extracted from the general series in (17). The solution in (19) and (20) have a simple form and they have been expressed completely as combinations of polynomials and algebraic functions. This has the advantage of simplifying the investigation into the physical properties of a dense anisotropic star.

Discussion

We have found solutions to the Einstein field equations for an anisotropic matter by utilizing the method of Frobenius for an infinite series; a particular form for one of the gravitational potentials was assumed and the anisotropic factor was specified. These solutions are given in terms of special functions and hypergeometric functions. For particular values of the parameters involved it is possible to write the solutions in terms of elementary functions: polynomials and product of polynomials and algebraic functions. The anisotropic factor may vanish in the solutions and we can regain the isotropic solutions. Thus our approach has the advantage of necessarily containing an isotropic neutral stellar solution found previously.

References

Chaisi, M & Maharaj, SD 2005, 'Compact Anisotropic Spheres with Prescribed Energy Density', *General Relativity and Gravitation*, vol. 37, pp. 1177-1189.

Dev, K & Gleiser, M 2003, 'Anisotropic Stars Instability', *General Relativity and Gravitation*, vol. 35, pp. 1435-1457.

Durgapal, M, C & Bannerji, R 1983, 'New Analytical Stellar Model in General Relativity', *Physical Review D*, vol. 27, pp. 328-331.

Maharaj, S, D & Komathiraj, K 2007, ‘Generalised Compact Spheres in Electric Fields’, *Classical and Quantum Gravity*’, vol. 24, pp. 4513-4524.

Mak, MK & Harko, T 2002, ‘An Exact Anisotropic Quark Star Model, *Chinese Journal of Astronomy and Astrophysics*, vol. 2, pp. 248-259.

Sharma, R & Mukherjee, S 2002, ‘Compact Stars A Core-Envelope Model, Modern Physics Letters A, vol. 17, pp. 2535-2544.