AN APPROXIMATION TECHNIQUE OF DEFINITE INTEGRAL USING SECOND ORDER TAYLOR POLYNOMIAL

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ABSTRACT

Numerical integration plays one of the most important roles in Applied Mathematics. The goal of the numerical integration is finding a better approximation value of a definite integral using numerical techniques which is highly challengeable. Numerous methods have been proposed in the literature to compute numerical integration. In this paper, we propose a better approach using second order Taylor polynomial to estimate the definite integrals and compare the accuracy of our method with different procedures available in the literature using an example. Also, we drive an upper bound estimation for the error. It is observed and illustrated that the proposed study provides more accurate results compared to the existing approaches.

Keywords: definite integral, error upper bound, numerical integration, Taylor polynomial

1. INTRODUCTION

Numerical integration is a process or technique of how the approximate numerical value of a definite integral can be found (Ullah, 2015). It has been successfully applied to study problems in fields of Mathematics, Engineering, Physical Sciences and Computer Science. The term *integral* may also denote the notion of anti-derivative, a function F whose derivative is the given function f.

Many mathematics researchers already have done broad investigations in the field of numerical integration. The purpose of this paper is to give an alternative method to find a better approximate value of definite integrals.

A definite integral is defined as a limit of Riemann sums and any Riemann sum could be used as an approximation to the integral (Kaw Autar, Keteltas Michael, 2012).

Let us consider a function $f(x)$ defined on a closed interval $[a, b]$. Divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$.

We choose $x_0(= a), x_1, x_2, \dots \dots, x_n(= b)$ be the endpoints of these subintervals. The definite integral of $f(x)$ from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

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provided that the limit exists, where $x_i{}^*$ is any point in the i^{th} subinterval $[x_{i-1},\ x_i]$, $i\ =\ 1, 2, \dots \dots\ n$ (Stewart, 1999). If the limit exists, f is said to be integrable on $[a, b]$.

2. METHODOLOGY

We will define an approximation function $P_2(x)$ on the interval $[x_i,\ x_{i+1}]$ by,

$$P_2(x) = f''(x_i)\frac{(x - x_i)^2}{2} + f'(x_i)(x - x_i) + f(x_i), \tag{1}$$

and hence,

$$\int_{x_i}^{x_{i+1}} P_2(x)dx = \int_{x_i}^{x_{i+1}} \left[f''(x_i)\frac{(x - x_i)^2}{2} + f'(x_i)(x - x_i) + f(x_i)\ \right]dx.$$

(Peer, 2015).

We shall drive a formula for the approximation of the integral $\int_a^b f(x)dx$ using midpoints of the subintervals of the interval $[a, b]$. Let $\bar{x}_i$ be the midpoint of $[x_i,\ x_{i+1}]$. Then,

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left[f''(\bar{x}_i)\frac{(x - \bar{x}_i)^2}{2} + f'(\bar{x}_i)(x - \bar{x}_i) + f(\bar{x}_i)\ \right]dx$$

$$= \sum_{i=0}^{n-1} \left[\frac{1}{2}f''(\bar{x}_i)\frac{(x - \bar{x}_i)^3}{3} + f'(\bar{x}_i)\frac{(x - \bar{x}_i)^2}{2} + f(\bar{x}_i)x\right]_{x_i}^{x_{i+1}}.$$

Substituting $\bar{x}_i = \frac{1}{2}(x_{i+1} + x_i)$, we get the left hand side is

$$= \sum_{i=0}^{n-1} \left\{\frac{1}{6}f''(\bar{x}_i)\left[\frac{1}{4}(x_{i+1} - x_i)^3\right] + f(\bar{x}_i)(x_{i+1} - x_i)\ \right\}.$$

$$= \sum_{i=0}^{n-1} \left[\frac{1}{24}f''(\bar{x}_i)(\Delta x)^3 + f(\bar{x}_i)(\Delta x)\right].$$

Thus,

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} \left[\Delta x \left(\frac{1}{24}f''(\bar{x}_i)(\Delta x)^2 + f(\bar{x}_i)\right)\right].$$

3. ERROR BOUND

In this section, let us prove two theorems related to the upper bound for error that occur due to this approximation.

Theorem 1 *Let the given function f be continuous over the interval $[a,b]$ and n be the number of subintervals which partitions the interval $[a,b]$. Let $\xi_0 \in [a,b]$ be such that*

$$|f(\xi_0)| = \max_{a\le\xi\le b}|f(\xi)|$$

Then the error E_T, for the Second Degree Taylor Polynomial Approximation of Line Integral satisfies

$$|E_T| \le \frac{(b-a)^4}{24n^3}\left|f^{(3)}(\xi_0)\right|.$$

Proof: Let us assume that the function f is continuously differentiable (at least third derivative exist) on (a,b). Let E_i be the error of i^{th} interval then total error E_T is sum of E_i's (Kaw Autar, Keteltas Michael, 2012).
Thus

$$E_T = \sum_{i=0}^{n-1} E_i\,.$$

Let us write

$$f(x) = P_2(x) + R_i(x)$$

where $R_i(x)$ is the remainder of the second degree Taylor approximation of f.
Then, $R_i(x) = \frac{f^{(3)}(\xi_i)}{3!}(x - x_i)^3$ for some $\xi_i \in [x_i, x]$ (Heinbockel, 2004).
Assume f''' is continuous on $[a,b]$. Then by intermediate value theorem (Heinbockel, 2004), there exist $\xi_i \in [a,b]$ such that,

$$\left|f^{(3)}(\xi_0)\right| = \max_{a\le\xi\le b}\left|f^{(3)}(\xi)\right| \ge \left|f^{(3)}(\xi_i)\right|, \qquad i = 0,1,2,\ldots\ldots,n-1.$$

Therefore,

$$|R_i(x)| \le \frac{f^{(3)}(\xi_0)}{3!}(x - x_i)^3 \qquad i = 0,1,2,\ldots\ldots,n-1.$$

Thus,

$$|E_i| = \left|\int_{x_i}^{x_{i+1}} [f(x) - P_2(x)]dx\right| \le \int_{x_i}^{x_{i+1}} |f(x) - P_2(x)|\,.$$

$$= \int_{x_i}^{x_{i+1}} |R_i(x)|\,dx\,.$$

$$\le \frac{\left|f^{(3)}(\xi_0)\right|}{3!}\int_{x_i}^{x_{i+1}} (x - x_i)^3\,dx\,.$$

$$= \frac{\left|f^{(3)}(\xi_0)\right|}{24}(x_{i+1} - x_i)^4 .$$
$$= \frac{\left|f^{(3)}(\xi_0)\right|}{24}(\Delta x)^4 .$$

This implies that,

$$|E_i| \leq \frac{\left|f^{(3)}(\xi_0)\right|}{24}(\Delta x)^4.$$

Hence, the total error is

$$|E_T| = \left|\sum_{i=0}^{n-1} E_i\right|.$$
$$\leq \sum_{i=0}^{n-1} \frac{\left|f^{(3)}(\xi_0)\right|}{24}(\Delta x)^4.$$
$$\leq \frac{(b-a)^4}{24n^3}\left|f^{(3)}(\xi_0)\right|.$$
$$|E_T| \leq \frac{(b-a)^4}{24n^3}\left|f^{(3)}(\xi_0)\right|.$$

Theorem 2 *Let* $f \in C^3[a,b]$ $h = \frac{(b-a)}{n}$ *, and* $x_i = a + ih$ *, for each* $i = 0,1,\dots,n$*. There exists a* $\xi \in [a,b]$ *for which the* $P_2(x)$ *Approximation Using Midpoints for* n *subintervals can be written with its error term as*

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1}\left[h\left(\frac{1}{24}f''(\bar{x}_i)h^2 + f(\bar{x}_i)\right)\right] + \frac{(b-a)}{24}h^3 f^{(3)}(\xi)$$

Proof:

Let $f \in C^3[a,b]$, $h = \frac{(b-a)}{n}$, and $x_i = a + ih$, for each $i = 0,1,\dots,n$. There exists a $\xi \in [a,b]$ such that

$$\int_{a=x_0}^{b=x_n} f(x)dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} (P_2 + R_2)dx.$$
$$= \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} P_2 dx + \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} R_2 dx .$$
$$= \sum_{i=0}^{n-1}\left[h\left(\frac{1}{24}f''(\bar{x}_i)h^2 + f(\bar{x}_i)\right)\right] + \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \frac{1}{6}(x - x_i)^3 f^{(3)}(\xi)dx .$$
$$= \sum_{i=0}^{n-1}\left[h\left(\frac{1}{24}f''(\bar{x}_i)h^2 + f(\bar{x}_i)\right)\right] + \sum_{i=0}^{n-1}\frac{1}{24}(x - x_i)^4 f^{(3)}(\xi) .$$

$$= \sum_{i=0}^{n-1}\left[h\left(\frac{1}{24}f''(\bar{x}_i)h^2 + f(\bar{x}_i)\right)\right] + \sum_{i=0}^{n-1}\frac{1}{24}h^4 f^{(3)}(\xi).$$

$$= \sum_{i=0}^{n-1}\left[h\left(\frac{1}{24}f''(\bar{x}_i)h^2 + f(\bar{x}_i)\right)\right] + \frac{1}{24}h^4\sum_{i=0}^{n-1} f^{(3)}(\xi).$$

$$= \sum_{i=0}^{n-1}\left[h\left(\frac{1}{24}f''(\bar{x}_i)h^2 + f(\bar{x}_i)\right)\right] + \frac{1}{24}h^4 n f^{(3)}(\xi).$$

$$= \sum_{i=0}^{n-1}\left[h\left(\frac{1}{24}f''(\bar{x}_i)h^2 + f(\bar{x}_i)\right)\right] + \frac{1}{24}h^3(b-a) f^{(3)}(\xi).$$

Hence,

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1}\left[h\left(\frac{1}{24}f''(\bar{x}_i)h^2 + f(\bar{x}_i)\right)\right] + \frac{(b-a)}{24}h^3 f^{(3)}(\xi).$$

4. RESULTS

We show the results of our Propose Approximation Method (P.A.M) compared with the Tangent Line Approximation (Tangent), Midpoint Rule (Mid), Trapezoidal Rule (Trapz) and Simpson's Rule (Simps) as n increases. The results are tabulated in table 4.1

Table 4.1 Comparison of $\int_{0.1}^{2.5}(3\log x + 2x^2 - \sin x)dx$ for $n = 5, n = 10, n = 50$ and $n = 100$.

	n - subintervals							
	n = 5		n = 10		n = 50		n = 100	
	Value	Error	Value	Error	Value	Error	Value	Error
Trapz	8.7744	0.2084	8.9139	0.0689	8.9795	0.0033	8.9820	8.3266e-04
Simp	3.9809	5.0019	8.9603	0.0225	8.9827	1.3976e-04	8.9828	1.0280e-05
Mid	9.0533	0.0705	9.0126	0.0298	8.9844	0.0016	8.9832	4.1531e-04
Tang	10.5761	1.5933	9.2973	0.3145	8.9909	0.0081	8.9847	0.0018
P.A.M	9.0075	0.0247	8.9868	0.0040	8.9828	1.5196e-05	8.9828	1.0126e-06

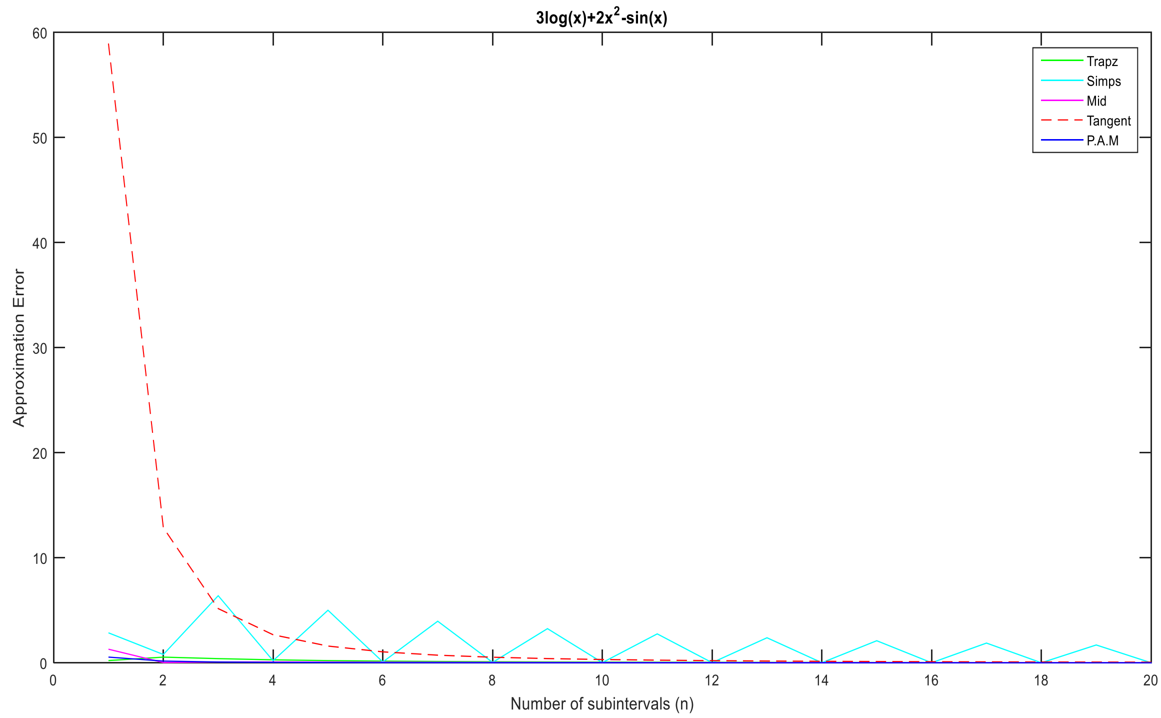

Figure 4.1 Graph of the approximate error for given in Table 4.1 as the number of subintervals n increases.

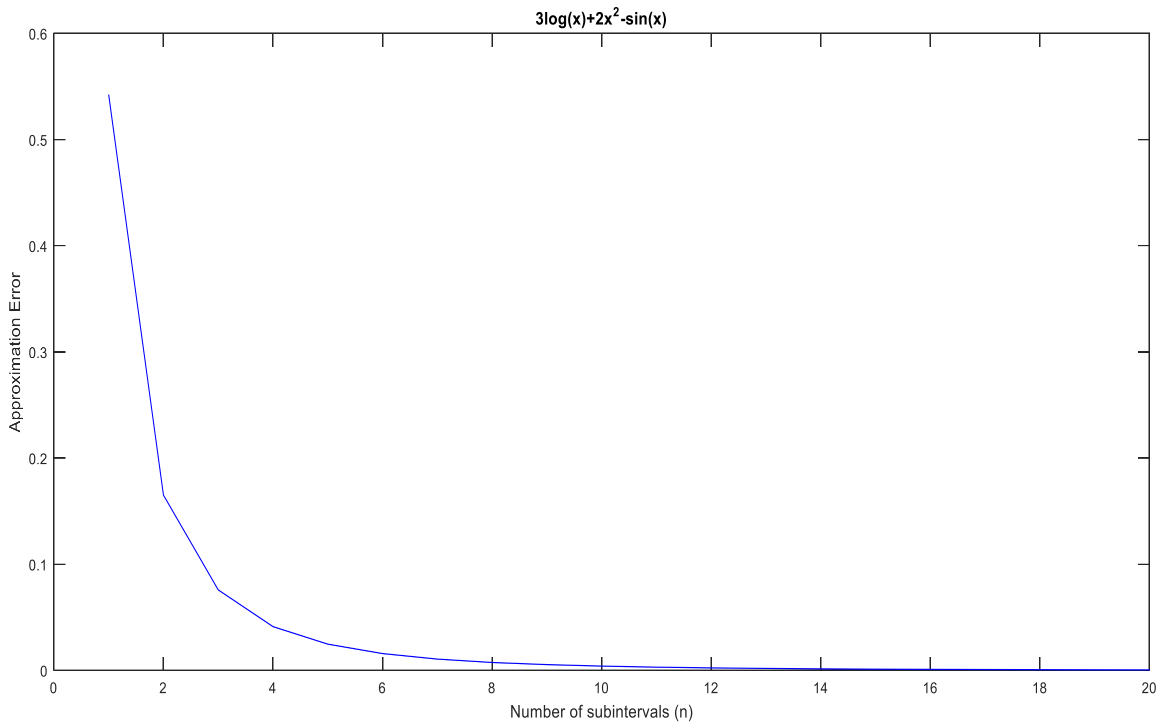

Figure 4.2 Graph of the approximate error for P.A.M as the number of subintervals n increases.

5. CONCLUSION

In this paper, we proposed a formula for approximating definite line integrals. Our approximation formula gives more efficient results in the point of accuracy compared to some of existing methods.

6. REFERENCES

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