

Solving First Order ODE With Initial Conditions Exactly Using Laplace Transform On MATLAB

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Abstract

The solution method for first-order ordinary differential equations (ODEs) with constant coefficients and initial conditions is presented in this study and is based on MATLAB. The suggested method depends on Laplace transforms to find exact results. The research introduces a special MATLAB function that is intended to precisely compute the exact solutions of certain ODEs while also offering other details like elapsed time and relevant figures. This method presents a systematic approach to handle these kinds of ODEs, solving the difficulties brought on by constant coefficients and initial conditions. It does this by using Laplace transformations. This method is expected to be used in disciplines like engineering and physics where second-order ODEs are frequent and exact solutions to them are important.

Keywords: Ordinary Differential Equations, Laplace Transforms, MATLAB

1. Introduction

Laplace transforms are widely used in a variety of fields because they may make complicated differential equations issues simpler. The analysis and solution of linear time-invariant systems in control theory, electrical circuits, and communication systems rely heavily on Laplace transforms in engineering. They make it easier for engineers to transform complex differential equations into algebraic equations, allowing them to create reliable systems.

Laplace transforms are used in physics to help solve differential equations that come up in the fields of electromagnetism, quantum mechanics, and classical mechanics. They enable physicists to effectively explain and anticipate complicated physical processes by providing insights into the behavior of dynamic systems. Additionally, Laplace transforms are essential for signal processing since they help with signal analysis and manipulation.

Laplace transforms help solve some issues and produce elegant solutions by bridging the gap between differential equations and algebraic equations in mathematics. Laplace transforms are very helpful in mathematical analysis for resolving boundary value issues, particularly ones with changing coefficients.

In conclusion, Laplace transforms have numerous and profound applications. They are necessary in disciplines like engineering, physics, signal processing, and mathematics due to their ability to transform complex differential equations into simpler algebraic forms. These transformations enable researchers and practitioners to effectively address difficult issues, resulting in improvements in technology, scientific knowledge, and problem-solving approaches.

In this paper, we consider first -order ordinary differential equations with constant coefficients and initial conditions.

$$a \frac{du(t)}{dt} + bu(t) = f(t)$$

$$u(0) = \varphi(t)$$

Where a and b are constant coefficients and $f(t)$ is the given function.

The MathWorks Company developed MATLAB, which stands for "MATrix LABoratory," a potent programming language and interactive environment. In many different industries, including engineering, research, and finance, it is frequently used. In terms of numerical computing, data analysis, and visualization, MATLAB is excellent. It offers a vast array of integrated tools, libraries, and functions that make it easier to complete tasks like statistical analysis, algorithm building, and matrix manipulation. Researchers, engineers, and students can use MATLAB to solve complicated problems, run simulations, and produce visual representations of data because of its user-friendly interface, comprehensive documentation, and active user community. Its widespread acceptance is facilitated by its adaptability and effectiveness.

2. Literature review

The Laplace transform of a function $f(t)$ defined $\forall t \geq 0$ is denoted by $F(s) = \mathcal{L}\{f(t)\}$ and defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt \quad ; \quad \text{where } s = a + ib, a, b \in \mathbb{R}$$

Applying the definition for first-order derivative, we get

$$\mathcal{L}\left\{\frac{du}{dt}\right\} = sU(s) - u(0) \quad \text{And } \mathcal{L}\{u(t)\} = U(s)$$

Bawar Mohammed Faraj and Faraedoon Waly Ahmed [2] discussed how to solve second-order ordinary differential equations with initial conditions using MATLAB and Laplace transforms. It offers a MATLAB tool that may swiftly calculate the approximate and accurate solutions for these equations, saving mathematicians time. The time-saving advantages of MATLAB for researchers in numerous domains are highlighted by these authors.

The study by W.W. Guo [3] investigated the application of Laplace transforms in solving constant-coefficient linear ODEs with generic initial conditions, enhancing mathematical problem-solving for engineering applications. The use of the Laplace transform to resolve linear fractional differential equations is covered in the study by Saeed Kazem [5]. His method's efficiency in generating precise solutions for various types of equations with fractional derivatives is illustrated by several examples. The practical value of the Laplace transform in resolving these issues is highlighted throughout the paper.

Dr. Dinesh Verma's article [6] presented mathematical derivations and solutions for specific instances and discussed how Laplace transformations can be used to solve differential equations with variable coefficients. He highlights how these equations can be transformed to become simpler through the use of Laplace transformations.

3. Methodology

The advantage of using the Laplace transform to solve linear ordinary differential (ODE) problems is that it offers particular solutions without the need to first identify the general solution and then assess the arbitrary constants. By following these procedures, Laplace transformation can be applied to ordinary and partial differential equations. Starting with a differential equation with an initial condition is the first step. Then apply the Laplace transform to the equation's two sides resulting in the subsidiary equation. Finally, take the inverse of the Laplace transformation and simplify it.

Example 1

Consider the following IVP

$$\frac{du}{dt} + 3u = 2e^{-t}; \quad u(0) = 0$$

Step 1

Take Laplace transform both sides

$$\mathcal{L}\left\{\frac{du}{dt}\right\} + \mathcal{L}\{3u\} = \mathcal{L}\{2e^{-t}\}$$

Step 2

Using the Laplace rules for derivative in definition 1 where $\mathcal{L}\{u\} = U(s)$

$$\{sU(s) - u(0)\} + 3U(s) = \frac{2}{s+1}$$

$$(s+3)U(s) = \frac{2}{s+1}$$

$$U(s) = \frac{2}{(s+1)(s+3)} = \frac{1}{s+1} - \frac{1}{s+3}$$

Step 3

Taking inverse Laplace we get

$$u(t) = \mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+3}\right\} = e^{-t} - e^{-3t}$$

Then the exact solution is $u(t) = e^{-t} - e^{-3t}$

4. MATLAB Approach

We take into account the initial value problem (1) and the Laplace transform definition (1). The MATLAB function is displayed; in this application, the constant ODE coefficients a, b and c , as well as u_0 is initial condition $u(0)$, are used. After calling the `matlplace.m` function, the function $f(t)$ must be input as the independent variable t . In the editor window of MATLAB, the following MATLAB Techniques are defined.

```

function [a, b, u0] = matlaplace(a, b, u0)
    syms s t U
    f = input('Input function f=');
    tic
    F = laplace(f, t, s);
    Ut = s*U - u0;
    Sol = solve(a*Ut + b * U - F, U);
    disp('The exact solution by Laplace transformation is ')
    u = ilaplace(Sol, s, t);
    toc
    ezplot(u)
    grid on
    title('Figure of Exact Solution by Laplace Transform')
    xlabel('t'), ylabel('u(t)')
end

```

Save the program and Call the function with coefficients a, b and initial condition u0.it will find the exact solution, visualize the graph and provide elapsed time.

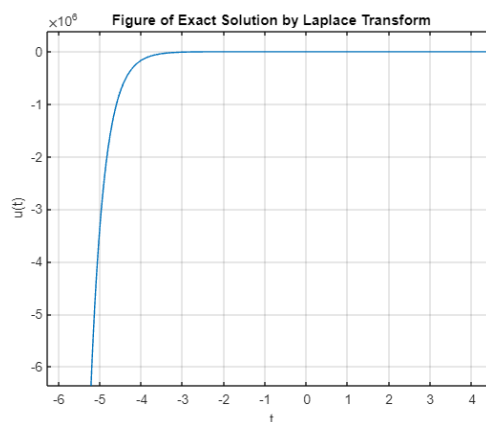
```
>>matlaplace(1,3,0)
```

Input function $f=2*\exp(-t)$

The exact solution by laplace transform is

$U=\exp(-t)-\exp(-3*t)$

Elapsed time is 0.101788



Consider some problems below

$$01.3 \frac{du}{dt} - 5u = e^t \sin(t) \cos(t); \quad u(0) = 0$$

It takes too long to figure out precisely using other programs or by hand and paper, but the Matlaplace function provides a figure of the solution in a matter of seconds.

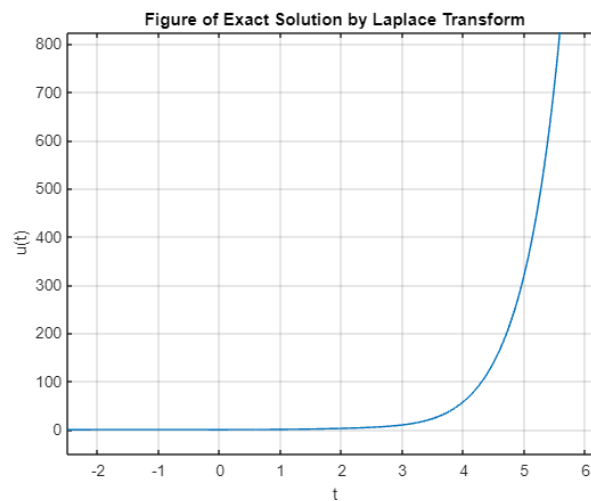
```
>>matlaplace(3,-5,0)
```

Input function $f = \exp(t) * \sin(t) * \cos(t)$

The exact solution by laplace transform is

$$U = (3 * \exp(5 * t / 3)) / 40 - (3 * \exp(t) * \cos(2 * t) + \sin(2 * t)) / 40$$

Elapsed time is 0.148697 seconds.



$$02. \frac{du}{dt} + 3u = \sin(2t) - \cos(t) - 1; \quad u(0) = -1$$

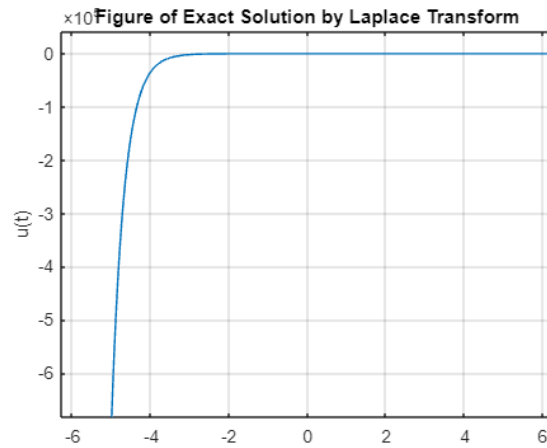
```
>>matlaplace(3,-5,0)
```

Input function $f = \sin(2 * t) - \cos(t) - 1$

The exact solution by Laplace transform is

$$U = (3 * \sin(2 * t)) / 13 - (83 * \exp(-3 * t)) / 390 - (2 * \cos(2 * t)) / 13 - (3 * \cos(t)) / 10 - \sin(t) / 10 - 1/3$$

Elapsed time is 0.464621 seconds.



$$03. u - \frac{4du}{dt} = \frac{\cosh(t) \sinh(t)}{e^{2t}} \quad u(0) = -2$$

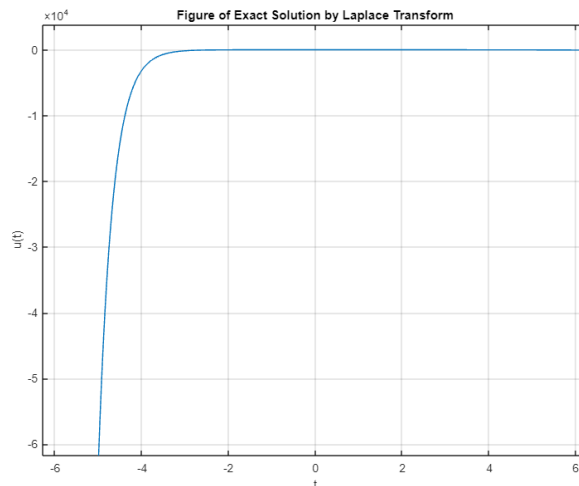
```
>>matlaplace(-4,1,-2)
```

Input function f=cosh(t)*sinh(t)/exp(t)

The exact solution by Laplace transform is

$$U = -\exp(3*t)/52 - (74*\exp(t/4))/39 - \exp(t)/12$$

Elapsed time is 0.090613seconds.



$$04. 10 \frac{du}{dt} = \sin^{10} t - e^{2t} - \cosh(t) \quad u(0) = 0$$

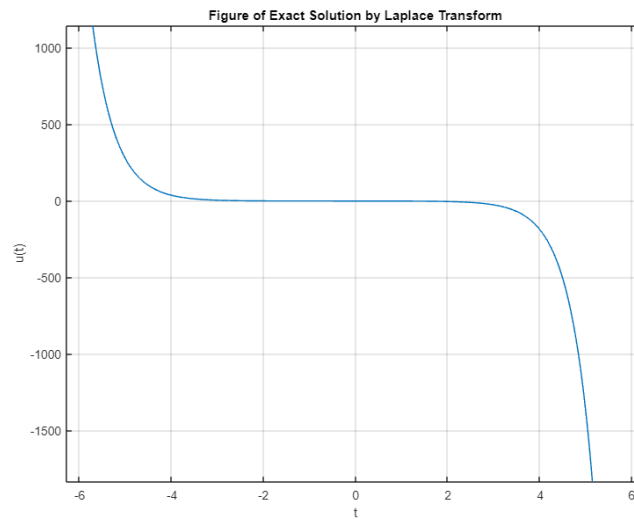
```
>>matlaplace(10,0,0)
```

Input function f=(sin(t))^10-(exp(t))^2-(cosh(t))^2

The exact solution by Laplace transform is

$$U = \exp(-2*t)/80 - (13*t)/512 - \exp(2*t)/16 - (21*\sin(2*t))/1024 + (3*\sin(4*t))/512 - (3*\sin(6*t))/2048 + \sin(8*t)/4096 - \sin(10*t)/51200 + 1/20$$

Elapsed time is 0.497554 seconds.



$$05. 3u - \frac{5du}{dt} = \sin^3 t - \frac{\cos(2t)}{t^3} \quad u(0) = -2$$

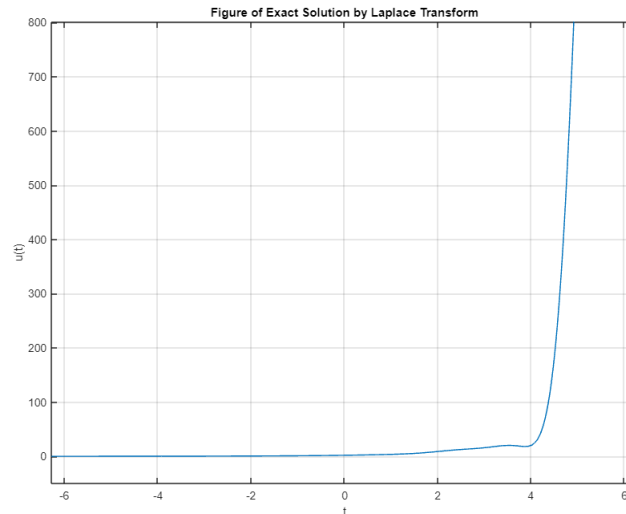
>>matlaplace(-5,3,2)

Input function f= exp(t)*sinh(t)*cos(2*t)*(sin(t))^2

The exact solution by Laplace transform is

$$U = (3*\cos(4*t))/3272 - (3*\cos(2*t))/436 + \exp(2*t)/56 + (122524520452*\exp((3*t)/5))/62632783101 + (5*\sin(2*t))/218 - (5*\sin(4*t))/818 - (7*\exp(2*t)*(cos(2*t) + (10*\sin(2*t))/7))/596 + (7*\exp(2*t)*(cos(4*t) + (20*\sin(4*t))/7))/3592 + 1/24$$

Elapsed time is 0.239795 seconds



5. Conclusions

For mathematicians, physicists, engineers, and scientists using Matlab software, regular calculations and applications take much less time. In this study, the Laplace transform is defined and used to solve an ODE example conventionally. A Matlab program is used to program the Laplace transform for initial value problems, and first-order ordinary differential equations with constant coefficients are constructed. The function estimates the exact solution in a matter of seconds, making it simple and helpful for researchers, and it also generates a graph of the exact solution. The direct simulation of the Laplace transformable equations using the Matlab program has advanced the field of research.

6. Future Work

Our upcoming research focuses on solving higher-order differential equations. Our goal is to develop methods and approaches for dealing with more complicated mathematical problems that involve higher derivatives, addressing challenges like increased equation complexity and limited analytical solutions, obtaining analytical solutions, handling heightened computational demands, and establishing intricate initial and boundary conditions.

References

1. De Hoog, Frank R., J. H. Knight, and A. N. Stokes. "An improved method for numerical inversion of Laplace transforms." *SIAM Journal on Scientific and Statistical Computing* 3.3 (1982): 357-366.
2. Faraj, Bawar Mohammed, and Faraedoon Waly Ahmed. "On the matlab technique by using laplace transform for solving second order ode with initial conditions exactly." *Matrix Science Mathematic* 3.2 (2019): 08-10.
3. Guo, William. "The Laplace transform as an alternative general method for solving linear ordinary differential equations." *STEM Education* 1.4 (2021): 309-329.

4. Honig, G., and U. Hirdes. "A method for the numerical inversion of Laplace transforms." *Journal of Computational and Applied Mathematics* 10.1 (1984): 113-132.
5. Kazem, Saeed. "Exact solution of some linear fractional differential equations by Laplace transform." *International Journal of nonlinear science* 16.1 (2013): 3-11
6. Verma, Dinesh. "Applications of laplace transformation for solving various differential equations with variable coefficients." *International Journal for Innovative Research in Science & Technology* 4.11 (2018): 124-127.