Analysis of Relations between Covariance Weights and Corresponding Maximum Eigen Values

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Abstract

This paper studies on how to get an idea through maximum eigen values, when allocating weights to covariance matrix. The eigen density distribution with respect to the largest eigen value is analysed. This study will help to determine the fluctuation of the eigen distribution with respect to allocated weight of the covariance matrix. This can be used to develop the classic portfolio asset allocation model by adding investors' ideas as parameters or weights. The maximum eigen value is 2.24 and the corresponding weight is 1.6, the minimum eigen value is 0.24 and the corresponding weight is 0.9. There are two peaks of the eigen values at 0.62 and 2.24 respectively 0.5 and 1.6 of weights. Two minimum points identified with corresponding eigen values are 0.43 and 0.24 respectively 0.2 and 0.9 of weights. For comparison the density function is plotted with Q = 3.22 and variance 0.85: this theoretical value was obtained assuming that the matrix is purely random except for its highest eigen value. The fact that the lower edge of the density is strictly positive (Except for the Q = 1); then there are no eigen values between 0 and the minimum eigen value.

Keywords: eigen values, maximum eigen values, eigen density, covariance, weights

Introduction

Since, early 1990s the market has witnessed a progressive shift, towards a more industrial view of the investment management process. There are several reasons for this change. Firstly, as a result of globalization, the universe of investable assets has grown many times over. Portfolio Managers might have to choose from among several thousand possible investments around the globe. In Modern Financial Theory there has been constant controversy about the concept of risk, and increasing interests in ways to measure it. This controversy has been accompanied by a growing investment industry in portfolio models based on sophisticated quantitative methods which require huge computing power. When the risk analysis of the portfolio is concerned, the empirical correlation matrices are very important. Even in the modern portfolio theory such as Markowitz portfolio optimization, the covariance or so called risk is minimized in order to find the optimum asset allocation.

Laloux, Cizeau and Potters (1990) have explained the financial correlations in their paper of "Random matrix theory and financial correlations". They find a remarkable agreement between the theoretical prediction (based on the assumption that the correlation matrix is random) and empirical data concerning the density of eigen values associated to the time series of the different stocks of the S&P500 (or other major markets). Daly and Rusking, (2007) in their paper of "Random matrix theory filters in Portfolio Optimization" have considered both equally and exponentially weighted covariance matrices and observed that the overall best method out-of-the sample was that of exponentially weighted covariance, with their Krzanowski stability-based filter applied to the correlation matrix. Pafca, Potters and Kondor (2004) have introduced a covariance matrix estimator in the paper of "Exponential weighting and random-matrix theory-based filtering of financial covariance matrices for portfolio optimization". They have calculated a spectrum of large exponentially weighted random matrices (whose upper band edge needs to be known for the implementation of the estimation) analytically, by a procedure analogous to that used for standard random matrices.

The main objective of this research to develop an asset allocation model among large number of investment channels by identifying the relation between the large eigen values and the allocated weights (decay factor). Also the relation between the eigen density distribution with respect to the eigen values.

Methodology

Average return$R_p = \sum_{r=1}^{n} p_i R_t$, where p_i (i= 1, 2, 3…..N) is the amount of capital investment in the ith asset and R_p are the expected return of the individual assets. The risk of the portfolio can be defined plugging the asset allocation component to the

variance and be formulated as $\sigma^2{}_p = \sum_{i,j=1}^{N} p_i\, C_{ij} p_j$ where C is the covariance matrix. There are a number of methods published to determine the optimal asset allocation satisfying maximizing the return and simultaneously minimizing the risk.
In the composition of the least risky portfolio a large weight is allocated on the eigenvector of C with the smallest eigen values.
For any N number of assets there are N(N-1)/2 entries of covariance elements. If the time length so called T is not very large compared to N, the determination of the covariance is noisy.

Random matrix theory

The empirical correlation matrix C is constructed from the time series of price changes $\delta x_i(t)$ where i is the asset and t the time. Then the equation is

$$C_{ij} = \frac{1}{T}\sum_{t=1}^{T} \delta x_i(t)\delta x_j(t) \tag{1}$$

Then the above equation can be written in the symbolical form as $\mathrm{C} = \frac{1}{\mathrm{T}}\mathrm{MM}^{\mathrm{T}}$ where M is an NxT rectangular matrix and M^{T} denotes matrix transformation of M. The null hypothesis of independent assets, which we consider now, translates itself in the assumption that the coefficients $M_{it} = \delta x_i(t)$ are independent, identically distributed, random variables, the so-called random Wishart matrices or Laguerre ensemble of the Random Matrix theory.

Then the eigenvalues density of the covariance matrix (C) is denoted by $pc(\lambda)$.

$$pc(\lambda) = \frac{1}{N}\frac{dn(\lambda)}{d(\lambda)} \tag{2}$$

Where $n(\lambda)$ is the number of eigenvalues of C less thanλ.
If M is a TxN random matrix, $pc(\lambda)$ is self-averaging and exactly known in the limit $N \to \infty, T \to \infty$ and $Q = \frac{T}{N} \geq 1$ fixed. Then

$$pc(\lambda) = \frac{Q}{2\pi\sigma^2}\frac{\sqrt{(\lambda_{max} - \lambda)(\lambda - \lambda_{\min})}}{\lambda} \tag{3}$$

Where $\lambda_{min}^{max} = \sigma^2\left(1 + \frac{1}{Q} \pm 2\sqrt{\frac{1}{Q}}\right)$

With $\lambda \epsilon [\lambda_{\min}, \lambda_{max}]$ and where σ^2 is equal to the variance of the elements of M, equal to 1 with our normalization. In the limit Q=1 the normalised eigen values.

Financial correlation and covariance matrices can be expressed, in general, in the form given by $\mathrm{C} = \frac{1}{\mathrm{T}}\mathrm{MM}^{\mathrm{T}}$, so matrices for historical data can be compared to those generalized from random returns. Here we defined the covariance matrix $V = \{\sigma_{ij}\}_{i,j=1}^{N}$ of returns by

$$\sigma_{ij} = \langle G_i(t)G_j(t)\rangle - \langle G_i(t)\rangle\langle G_j(t)\rangle \tag{4}$$

Where $\langle . \rangle$ refers to the mean over time and the correlation matrix$C = \{\rho_{ij}\}_{i,j=1}^{N}$ is given by

$$\rho_{ij} = \sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}} \tag{5}$$

Where $\{\boldsymbol{G}_i(\boldsymbol{t})\}_{t=1......T}^{i=1.....N}$ are the returns

$G_i(t) = \ln\left(\frac{S_i(t)}{S_i(t-1)}\right)$ and where $S_i(t)$ is the spot price of asset I at time t.

Random matrix theory and exponentially weighted covariance

In extending RMT filtering to exponentially weighted matrices, Pafca (2004) has analyzed matrices of the form $M = \{m_{ij}\}_{i,j=1}^{N}$ with

$$m_{ij} = \sum_{k=0}^{\infty} (1-\alpha)\alpha^k x_{ik} x_{jk} \tag{6}$$

And where $\{x_{ik}\}_{k=0\ldots\infty}^{i=1\ldots N}$ are assumed to be N.I.D. $(0, \sigma^2)$.They have shown that in the special case $N \to \infty, \alpha \to 1$ with $Q = \frac{1}{N(1-\alpha)}$ fixed, the density $pc(\lambda)$, of the eigenvalues of M is given by

$pc(\lambda) = \frac{Qv}{\pi}$ where v is the root of

$$F(v) = \frac{\lambda}{\sigma^2} - \frac{v\lambda}{\tan(v\lambda)} + \ln(v\sigma^2) - \ln(\sin(v\lambda)) - \frac{1}{Q} \tag{7}$$

$F(v)$is well defined on the open interval (0, π/λ). If a root doesn't exist on this given interval for a given value of λ, we define $\mathrm{pc}(\lambda) = 0$ for that λ. The family of matrices, defined by

$\mathrm{m_{ij}} = \sum_{\mathrm{k}=0}^{\infty}(1-\alpha)\alpha^{\mathrm{k}}\mathrm{x_{ik}}\mathrm{x_{jk}}$, includes the Risk metrics, covariance and correlation matrices. Following this, we define the exponentially weighted covariance matrix $V^* = \{\sigma_{ij}^*\}_{i,j=1}^{N}$ by

$$\sigma_{ij}^* = \frac{1-\alpha}{1-\alpha^T} \sum_{t=0}^{T-1} \alpha^t (G_t(T-t) - \langle G_i(t)\rangle)(G_j(T-t) - \langle G_j(t)\rangle) \tag{8}$$

And define the corresponding, exponentially weighted, correlation matrix $C^* = \{\rho_{ij}^*\}_{i,j=1}^{N}$ by $\rho_{ij}^* = \sigma_{ij}^* / \sqrt{\sigma_{ii}^* \sigma_{jj}^*}$.

Here α is commonly called the decay factor. Maximum Eigenvalue of an Exponentially Weighted Random Matrix can be found using

$F(v) = \frac{\lambda}{\sigma^2} - \frac{v\lambda}{\tan(v\lambda)} + \ln(v\sigma^2) - \ln(\sin(v\lambda)) - \frac{1}{Q}$, but a more efficient method can be derived as follows. On the interval v ε(0, π/λ). The following limits hold

$$\lim_{v\to\infty} F(v) = \frac{\lambda}{\sigma^2} - \ln\left(\frac{\lambda}{\sigma^2}\right) - \frac{1}{Q} - 1 \tag{9}$$

$$\lim_{v\to\pi/\lambda} F(v) = \infty$$

Moreover, F(v) is increasing on the interval v ε(0, π/λ) since for x=vλ

$$F'(v) = \frac{1}{v} - \frac{x}{v}\left(\frac{2\tan(x) - xsec^2(x)}{tan^2(x)}\right)$$

$$F'(v) = \frac{1}{v}\left(\frac{2\sin^2(x) - 2xsin(x)\cos(x) + x^2}{\sin^2(x)}\right) \tag{10}$$

And also $h(x) = \sin^2(x) - 2xsin(x)\cos(x) + x^2 > 0$ on xε(0,π) which is true because h(0)=0 and $h'(x) = 4x\sin^2(x) > 0$. Therefore, a root of F(v) exists on v ε(0, π/λ) for a given Q and λ when its lower limit is negative on the interval, i.e. When

$$\frac{\lambda}{\sigma^2} - \ln\left(\frac{\lambda}{\sigma^2}\right) < \frac{1}{Q} + 1 \tag{11}$$

Now $\frac{\lambda}{\sigma^2} - \ln\left(\frac{\lambda}{\sigma^2}\right) \geq 1$, with a minimum at 1 when $\lambda = \sigma^2$ and

$$\frac{1}{Q} + 1 > 1 \tag{12}$$

Thus, the theoretical maximum eigenvalue is the solution of

$$\frac{\lambda}{\sigma^2} - \ln\left(\frac{\lambda}{\sigma^2}\right) = \frac{1}{Q} + 1 \tag{13}$$

Algorithm

In order to get the maximum eigen values of the covariance matrix with respect to the different weights of the covariance matrix, the following algorithm is used.

%calculate the risk metrics "Technical Document" (1996) exponentially

%weighted covariance matrix, correlation and volatilities.

%Inputs:

%data- needs to be in format TxK with T=# observations, k=# assets

%alpha=decay factor

1. Inpute: [r,c] = size(data) : needs to be in format TxK with T=# observations, k=# assets
2. Input: alpha=decay factor
3. data_mwb = data-replicate the matrix (mean(data,1),r,1);
4. alphavec = alpha.^(0:1:r-1)';
5. data_tilde= replicate matrix (sqrt(alphavec),1,c).*data_mwb;
6. cov_ewma = 1/sum(alphavec)*(data_tilde'*data_tilde);
7. corr_ewma = zeros(c)
8. vol_ewma=zeros(c,1);
9. for i = 1:c
10. for j=1:c

11. corr_ewma(i,j) = cov_ewma(i,j)/sqrt(cov_ewma(i,i)*cov_ewma(j,j));
12. end
13. vola_ewma(i) = sqrt(cov_ewma(i,i));
14. end

Results & Discussion

The data set used to test this research is SGX (Singapore Exchange Market) stocks, with index composition taken as of 17/04/2014. The data set runs from 06/11/2008 to 04/04/2014 (1339 days) and any series not covering the entire period were discarded, leaving a total 406 stocks.

Table 1: Maximum eigen values with weights

	Maximum Eigen values					
Weight	Rank1	Rank2	Rank3	Rank4	Rank5	Rank6
0	0.4828	0	0	0	0	0
0.1	0.4384	0.2171	0.0047	0.0005	0.0001	0
0.2	0.426	0.3531	0.0167	0.0032	0.0014	0.0003
0.3	0.5239	0.33	0.0333	0.0095	0.0062	0.002
0.4	0.5947	0.2852	0.0516	0.0192	0.0169	0.0071
0.5	0.6189	0.2387	0.0687	0.0355	0.03	0.0179
0.6	0.5954	0.1919	0.0816	0.0594	0.0412	0.0356
0.7	0.5243	0.1456	0.0896	0.0815	0.0576	0.049
0.8	0.405	0.1041	0.0993	0.0796	0.073	0.0524
0.9	0.2359	0.0912	0.0744	0.0656	0.0555	0.0523
1	0.5768	0.3425	0.2026	0.1528	0.0917	0.0854
1.1	0.6746	0.5019	0.4339	0.2956	0.2427	0.2112
1.2	1.1035	0.6638	0.5983	0.3893	0.2421	0.2311
1.3	1.4571	0.7763	0.6331	0.4123	0.2329	0.2056
1.4	1.7561	0.863	0.607	0.4077	0.2177	0.1879
1.5	2.013	0.9225	0.5651	0.3891	0.2004	0.1602
1.6	2.2363	0.9615	0.5221	0.3627	0.1813	0.1329
1.7	0	0	0	0	0	0

The table 1 shows the largest six eigen values vs. weights of the covariance matrix. The above table has been calculated based on raw data of closing prices. As the first step of the calculation process, the covariance matrix is calculated based on the closing prices and then the weighted covariance matrix is calculated. Then for each weighted covariance matrix we calculate 6 largest eigen values (Rank 1 to Rank 6). These values are summarized in the above table. The minimum weight assigned is zero and the maximum weight is 1.7. When the exponential weights are more than 1.7, then the corresponding maximum eigen values are zeros. In order to analyse this phenomena further, these values are plotted as it is in fig. 1.

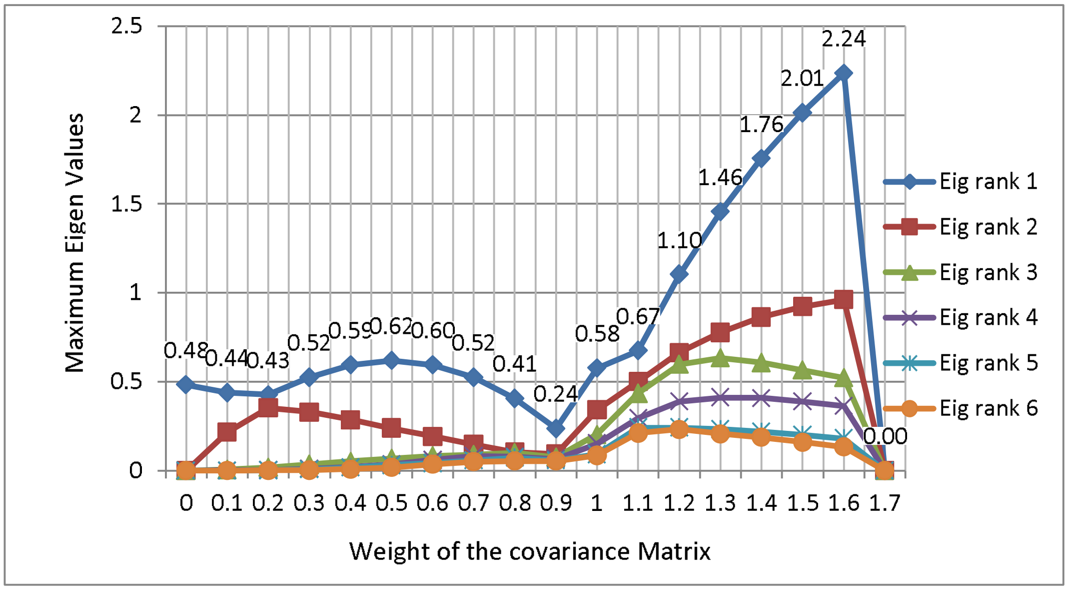

Figure 1: Maximum eigen values vs. weights of covariance

The figure 1 denotes a graphical representation of table 1. This clearly shows the effect of maximum eigen values when the weights are changed from 0 to 1.7. Fig. 2 represent the eigen density of the covariance matrix at the maximum eigen value which can be taken when the weight is set to 1.6.

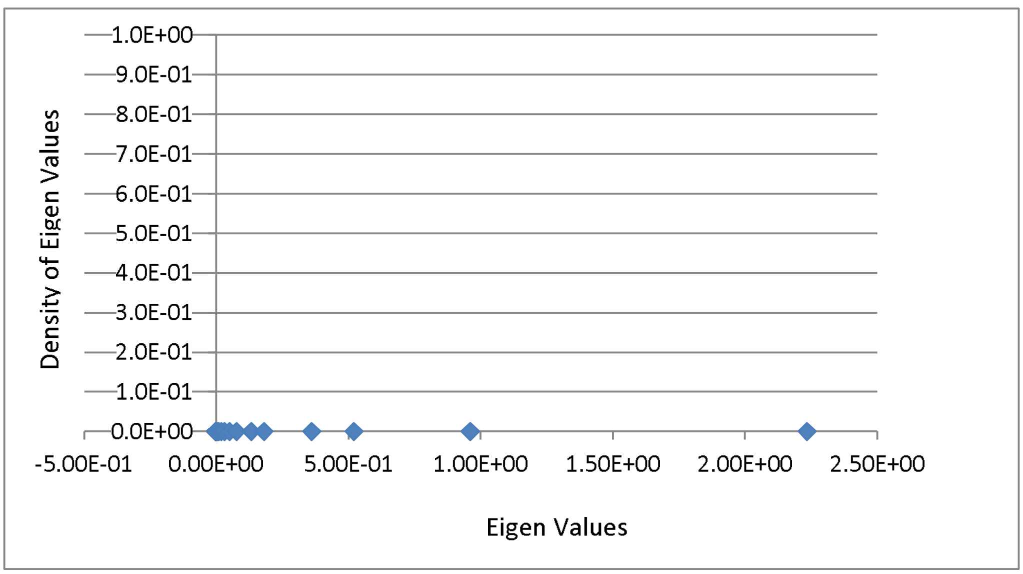

Figure 2: Eigen density vs. eigen values

Conclusions

The maximum eigen value is 2.24 and the corresponding weight is 1.6, and the minimum eigen value is 0.24 and the corresponding weight is 0.9. There are two peaks of the eigen values at 0.62 and 2.24 with respective weights of 0.5 and 1.6. The two minimum points that can be found hold corresponding eigen values at 0.43 and 0.24 with respective weights of 0.2 and 0.9. For comparison the density function is plotted with Q=3.22 and variance 0.85: this theoretical value was obtained assuming that the matrix is purely random. The lower edge of the density is strictly positive (Except for the Q=1); then there is no eigen values between 0 and the minimum eigen value. These results can be used to weight the covariance matrix with the feedback of maximum eigen values. The minimum possible weight and the maximum possible weight can be understood from the process. Then the weight can be effectively allocated while referring to the maximum eigen values. Thereafter the selected weight can be used to get the eigen density distribution. The process can thus be extended to get an idea about the controllable risk matrix with respect to the external factors such as investors' views towards to the risk.

References

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