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Contribution of Ridge Type Estimators in Regression Analysis

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Abstract

Regression Analysis is one of the most widely used statistical techniques for analyzing multifactor data. Its broad appeal results from the conceptually simple process of using an equation to express the relationship between a set of variables. Regression analysis is also interesting theoretically because of the elegant underlying mathematics. Successful use of regression analysis requires an appreciation of both the theory and the practical problems that often arise when the technique is employed with real world data.

In the model fitting process the most frequently applied and most popular estimation procedure is the Ordinary Least Square Estimation (OLSE). The significant advantage of OLSE is that it provides minimum variance unbiased linear estimates for the parameters in the linear regression model.

In many situations both experimental and non-experimental, the independent variables tend to be correlated among themselves. Then inter-correlation or multicollinearity among the independent variables is said to be exist. A variety of interrelated problems are created when multicollinearity exists. Specially, in the model building process, multicollinearity among the independent variables causes high variance (if OLSE is used) even though the estimators are still the minimum variance unbiased estimators in the class of linear unbiased estimators.

The main objective of this study is to show that the unbiased estimation does not mean good estimation when the regressors are correlated among themselves or multicollinearity exists. Instead, it is tried to motivate the use of biased estimation (Ridge type estimation) allowing small bias and having a low variance, which together can give a low mean square error.

This study also reveals the importance of the theoretical results already obtained, and gives a path for a researcher for the application of the theoretical results in practical situations.

Keywords*: *Multicollinearity, Least Square Estimation, Restricted Least Square Estimation, Modified Ridge Regression, Restricted Ridge Regression.

Introduction

The Problem of multicollinearity and its statistical consequences on a linear regression model are very well-known in statistics. The multicollinearity is defined as the existence of nearly linear dependency among the regressors in the linear model $Y = X\beta + \varepsilon$. The best way of explaining the existence of multicollinearity is to look at the correlation matrix of X'X, variance inflation factor (IVF), and conditional index number.

The existence of multicollinearity may result in wide confidence intervals for individual parameters (unstable estimates), may give estimates with wrong signs and may effect our decision in a hypothesis testing. Strong and Severe multicollinearity may make the estimates so unstable that they are practically useless. To overcome this, different remedial actions have been proposed. A popular numerical technique to deal with multicollinearity is that of ridge type regression.

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In literature several biased estimation procedures were introduced for solving the problem of multicollinearity. Among them the biased regression technique namely, Ridge Regression Estimation (RRE), was first introduced by Hoerl (1964), and further developed by Hoerl and Kennard (1970a, b). Restricted Ridge Regression Estimation (RRRE) introduced by Sarkar (1992), and Modified Ridge Regression Estimation (MRRE) introduced by Swindel (1976), respectively were frequently used biased estimation methods. These methods were rapidly developed in the recent years.

The primary aim in this paper is to compare ridge biased estimators, with the same ridge type estimators and unbiased estimators Ordinary Least Square Estimator (OLSE) and Restricted Least Square Estimator (RLSE) by using the scalar mean square error matrix (mse) and matrix mean square error matrix (MSE).

The Model and Estimators

We consider the standard multiple linear regression model in matrix form

$$Y = X\beta + \varepsilon, \tag{1}$$

where Y is an (nx1) vector of observations on the response (or dependent) variable, X is an (nxp) matrix of observations on p non stochastic explanatory variables with full column rank, β is a (nx1) vector of unknown parameters associated with the p regressors, and ε is a (nx1) vector of disturbances with expectation $E(\varepsilon) = 0$, and dispersion (variance-covariance) matrix $\mathbf{D}(\varepsilon) = \sigma^2 I_n$.

In the case of multicollinearity among the regressors, the Ordinary Least Squares Estimator (OLSE),

$$\hat{\beta} = S^{-1}X'Y \tag{2}$$

is not preferred for estimating the unknown parameters in β, where $S = X'X$.
Instead the method of ridge regression has been developed by Hoerl and Kennard (1970), who introduced the ridge regression estimator (RRE) as

$$\hat{\beta}_R = (S + \mathrm{k}I_p)^{-1}X'Y = W\beta, \tag{3}$$

where $W = (I_p + kS^{-1})^{-1}$. Note that when k=0, $\hat{\beta}_R = \hat{\beta}$.
If a set of q linear restrictions is available on β in the form

$$R\beta = r, \tag{4}$$

where the matrix R is a known (qxp) matrix with full row rank (q<p) and the vector r is a known (qx1) vector.

The restricted least squares estimator (RLSE) of β is another suitable estimator for handling collinear data, and defined as

$$\beta_r^* = + S^{-1}R'(RS^{-1}R')^{-1}(r - R\hat{\beta}). \tag{5}$$

Some other alternative bias estimators introduced in literature are
The restricted ridge regression estimator (RRRE) $\beta_{(k)}^*$ - Sarkar (1992)

$$\beta_{(k)}^* = W\beta_r^* \tag{6}$$

The modified ridge regression estimator (MRRE) $\mathbf{b}(k, b^*)$, based on a fixed vector b^* of prior estimate of β - Swindel (1976),

$$\mathbf{b}(k, b^*) = (S + kI_p)^{-1}(X'Y + kb^*)\ ; \quad k \geq 0 \tag{7}$$

Statistical Properties of The Estimators

The bias vector **B**(.), dispersion matrix **D**(.), scalar mean square error **mse**(.), and the matrix mean square error **MSE**(.) of OLSE, RLSE, RRE, RRRE, MRRE, LE and RLE are given below:

(i) Statistical properties of unbiased **OLSE**

$$\mathbf{B}(\hat{\beta}) = \mathbf{0} \qquad \mathbf{D}(\hat{\beta}) = \sigma^2(\mathbf{X'X})^{-1}, \tag{8}$$

$$\mathbf{mse}(\hat{\beta}) = \sigma^2 \mathrm{tr}(\mathbf{X'X})^{-1}, \tag{9}$$

and $$\mathbf{MSE}(\hat{\beta}) = \sigma^2(\mathbf{X'X})^{-1} \tag{10}$$

(ii) Statistical properties of unbiased **RLSE**

Letting $\mathbf{A} := \mathbf{S}^{-1} - \mathbf{S}^{-1}\mathbf{R}'(\mathbf{RS}^{-1}\mathbf{R}')^{-1}\mathbf{RS}^{-1}$ and $\delta := \mathbf{r} - \mathbf{R}\beta$ then

$$\mathbf{B}(\beta_r^*) = \mathbf{0} \qquad \mathbf{D}(\beta_r^*) = \sigma^2\mathbf{A}, \tag{11}$$

$$\mathbf{mse}(\beta_r^*) = \sigma^2\mathrm{tr}(\mathbf{A}) + \delta'(\mathbf{RS}^{-1}\mathbf{R}')^{-1}\mathbf{RS}^{-2}\mathbf{R}'(\mathbf{RS}^{-1}\mathbf{R}')^{-1}\delta \tag{12}$$

and $$\mathbf{MSE}(\beta_r^*) = \sigma^2\mathbf{A} + [\mathbf{S}^{-1}\mathbf{R}'(\mathbf{RS}^{-1}\mathbf{R}')^{-1}\delta][\mathbf{S}^{-1}\mathbf{R}'(\mathbf{RS}^{-1}\mathbf{R}')^{-1}\delta]' \tag{13}$$

(iii) Statistical properties of biased **RRE**

$$\mathbf{B}(\hat{\beta}_R) = k^2\beta'(\mathbf{X'X} + k\mathbf{I})^{-2}\beta \qquad \mathbf{D}(\hat{\beta}_R) = \sigma^2(\mathbf{X'X})(\mathbf{X'X} + k\mathbf{I})^{-2}, \tag{14}$$

$$\mathbf{mse}(\hat{\beta}_R) = \sigma^2\ \mathrm{tr}[(\mathbf{X'X} + k\mathbf{I})^{-1}(\mathbf{X'X})(\mathbf{X'X} + k\mathbf{I})^{-1}] + k^2\beta'(\mathbf{X'X} + k\mathbf{I})^{-2}\beta, \tag{15}$$

and $$\mathbf{MSE}(\hat{\beta}_R) = \sigma^2(\mathbf{X'X})(\mathbf{X'X} + k\mathbf{I})^{-2} + k^2(\mathbf{X'X} + k\mathbf{I})^{-1}\beta\beta'(\mathbf{X'X} + k\mathbf{I})^{-1} \tag{16}$$

(iv) Statistical properties of biased **RRRE**

$$\mathbf{B}(\beta_{(k)}^*) = \mathbf{W}S^{-1}\delta^* - kS(k)^{-1}\beta \qquad \mathbf{D}(\beta_{(k)}^*) = \sigma^2\mathbf{WAW'}, \tag{17}$$

$$\mathbf{mse}(\beta_{(k)}^*) = \sigma^2\mathrm{tr}(\mathbf{WAW'}) + k^2\beta' S(k)^{-2}\beta, \tag{18}$$

and $$\mathbf{MSE}(\beta_{(k)}^*) = S(k)^{-1}\,[\sigma^2(S - R'(RS^{-1}\mathbf{R}')^{-1}\mathbf{R}) + (k\beta - \delta^*)(k\beta - \delta^*)']\,S(k)^{-1} \tag{19}$$

where $\mathbf{A} = S^{-1} - S^{-1}R'(RS^{-1}R')^{-1}RS^{-1}$, $S(k) = S + kI_p$, $\delta := \mathbf{r} - \mathbf{R}\beta$, and $^* = R'(RS^{-1}\mathbf{R}')\delta$

(v) Statistical properties of biased **MRRE**

$$\mathbf{B}(\mathbf{b}(k, b^*)) = -\,kS(k)^{-1}(\beta - b^*) \qquad \mathbf{D}(\mathbf{b}(k, b^*)) = \sigma^2 S(k)^{-1}S\,S(k)^{-1} \tag{20}$$

$$\mathbf{mse}(\mathbf{b}(k, b^*)) = \mathrm{tr}[S(k)^{-1}[\sigma^2 S + k^2(\beta - b^*)(\beta - b^*)']S(k)^{-1} \tag{21}$$

and $$\mathbf{MSE}(\mathbf{b}(k, b^*)) = S(k)^{-1}\,[\sigma^2 S + k^2(\beta - b^*)(\beta - b^*)']S(k)^{-1} \tag{22}$$

Several statisticians compared the scalar mean square errors as well as matrix mean square errors of the above estimators, and obtained conditions for domination of one estimator by another estimator. These conditions are strictly theoretical, and are based on unknown parameters β, σ^2 and also on the choice of ridge parameter k in the RRE, RRRE and MRRE, and d in the LE and RLE.

Mean Square Error Matrix Comparison between the Estimators

When multicollinearity exist among the regressors the following necessary and sufficient mean square error matrix conditions must be satisfied for one estimator is better than the other estimator.

1. **Comparison between OLSE Vs RLSE**
   The necessary and sufficient conditions for the unbiased RLSE is better than the unbiased OLSE is:
   (i) MSE(OLSE) – MSE(RLSE) is n.n.d
   (ii) $(r - R\beta)'(RS^{-1}R')^{-1}(r - R\beta) \leq \sigma^2$

2. **Comparison between OLSE Vs RRE**
   The necessary and sufficient condition for the superiority of biased RRE over the unbiased OLSE is:
   $0 < k < \sigma^2 / \alpha^2_{max}$ where $\alpha = P'\beta$ and $X'X = P'\wedge P$

3. **Comparison between RRE VS RRRE**
   The necessary and sufficient condition for the superiority of biased RRRE over the biased RRE is:
   $(S + kI_p)^{-1}S^{-1}R'(RS^{-1}R')RS^{-1}(S + kI_p)^{-1}$ p.s.d matrix

4. **Comparison between RLSE VS RRRE**
   The necessary and sufficient conditions for the biased RRRE is better than the unbiased RLSE is:
   $0 < k < \sigma^2/\alpha^{*2}_{max}$ where $\alpha^{*2}_i = \alpha^2_i /(\lambda_i b_{ii})$, where $\alpha = P'\beta$.

5. **Comparison between MRRE Vs RRRE**
   The necessary and sufficient condition for the superiority of biased RRRE over the biased MRRE is:
   (i) $k^2 < (\sigma^2/\beta'S^{-1}\beta)$
   (ii) $[(\beta - b^*)' S^{-1}(\beta - b^*) + (\sigma^2/k^2)][\beta'S^{-1}\beta - (\sigma^2/k^2)] \leq [(\beta - b^*)' S^{-1}\beta]^2$

Results and Discussions

To derive the required estimators, five independent variables and a dependent variable are generated by using the following Monte Carlo equations.

$$X_{ij} = (1 - \rho^2)^{1/2} z_{ij} + \rho z_{i6} ,$$
$$Y_i = (1 - \rho^2)^{1/2} z_i + \rho z_{i6} , \quad i = 1, 2, 3, \ldots, 100 \quad j = 1, 2, 3, 4, 5 \qquad (21)$$

where z_i, z_{i1}, z_{i2}, z_{i3}, z_{i4}, z_{i5}, and z_{i6} are independent standard normal pseudo-random numbers and ρ is specified so that the correlation between any two explanatory variables is given by ρ^2. The number of observations are 100 in this case, and the explanatory variables are collinear with high correlation coefficients. These variables are then standardized so that $X'X$ is in a correlation form. Two different sets of correlations namely $\rho = 0.95$ and 0.99 are considered for this study.

Using correlation matrix **X'X,** variance inflation factors and condition indices it can be shown that the independent variables are strongly and severely correlated among them when $\rho = 0.95$ and 0.99, respectively.

The methods described to estimate the parameters *d* and *k* in section 3 were used and obtained the estimators OLSE, RLSE, RRE, RRRE, MRRE, LE and RLE.

The stochastic properties of the scalar mean squared error (**mse**) and matrix mean square error (**MSE**) of parameter vector were computed for the set of estimated coefficients, and compared them to demonstrate the effect on multicollinearity, and to select the appropriate estimator for the given situation.

Comparison between OLSE and RLSE

When the restrictions are true the RLSE estimator was obtained using the methods suggested in section three.

The mse parameter vector of OLSE and RLSE

The scalar mean square error of the parameter vector **(mse)** is calculated using the expressions given in section 3 (i) & 3 (ii) and it is given below Table 1.

The MSE parameter vector of OLSE and RLSE

When the restrictions are true the unbiased RLSE is better than the unbiased OLSE then MSE(OLSE) – MSE(RLSE) is non-negative definite matrix. This implies that the eigenvalues of the matrix MSE(OLSE) – MSE(RLSE) is greater than or equal to zero.

The eigenvalues of the matrix MSE(OLSE) – MSE(RLSE) are: [-6.8469e-008, 5.0432e-008, 1.1462e-007, 7.0672e-003, 1.1856e-002] and [-7.9947e-008, -3.7664e-008, 2.2041e-008, 9.0246e-003, 1.7320e-002] when $\rho = 0.95$ and $\rho = 0.99$ respectively.

If we consider the above eigenvalues three values are closed to zero (-6.8469e-008, 5.0432e-008, 1.1462e-007) and other rest of two values are positive when $\rho = 0.95$, similarly three values are closed to zero (-7.9947e-008, -3.7664e-008, 2.2041e-008) and two values are positive when $\rho = 0.99$. So from these eigenvalues results we can say that the matrix MSE(OLSE) – MSE(RRE) is a non negative definite matrix.

From the **mse** and **MSE** results we can say that the unbiased RLSE is superior over the unbiased OLSE when the restrictions are indeed correct.

Table 1: The scalar mean square error of OLSE and RLSE

	$\rho = 0.95$		$\rho = 0.99$	
	OLSE	**RLSE**	**OLSE**	**RLSE**
mse	0.05474	0.03581	0.05229	0.02594

In both situation when $\rho = 0.95$ and $\rho = 0.99$ the scalar mean square error (mse) of RLSE is smaller than the OLSE.

Comparison between OLSE and RRE

When ρ = 0.95 and 0.99, the ridge regression parameter *k* was found as 0.024 and 0.023 respectively, using the methods suggested in section 3.1 (2). Using these *k* values, the RRE was obtained.

The mse parameter vector of OLSE and RRE

The scalar mean square error of the parameter vector **(mse)** is calculated using the expressions given in section 3 (i) & 3 (iii) and it is given below Table 2.

All the above eigenvalues are positive but only one eigenvalue (-9.7931e-006) is closed to zero when ρ = 0.95, and also one eigenvalue (-7.6175e-006) is closed zero when ρ = 0.99. So it can be seen that matrix MSE(OLSE) – MSE(RRE) is a non negative definite matrix.

According to the **mse** and **MSE** results it was found that the biased RRE is superior over the unbiased OLSE when $0<k<0.12832$, and $0<k<0.02354$ for ρ = 0.95 and 0.99 respectively.

Table 2: The scalar mean square error of OLSE and RRE

	ρ = **0.95**		ρ = **0.99**	
	OLSE	**RRE**	**OLSE**	**RRE**
mse	0.05474	0.03857	0.05229	0.01901

When ρ = 0.95 and 0.99 the mse of RRE has smaller than the OLSE.

The MSE parameter vector of OLSE and RRE

The biased RRE is better than the unbiased OLSE when MSE(OLSE) – MSE(RRE) is non-negative definite matrix. It means the eigenvalues of MSE(OLSE) – MSE(RRE) is greater than or equal to zero.

The eigenvalues of the matrix MSE(OLSE) – MSE(RRE) are: [-9.7931e-006, 1.3527e-003, 3.2003e-003, 4.6277e-003, 6.9983e-003] and [-7.6175e-006, 5.8436e-003, 6.6398e-003, 8.7361e-003, 1.2060e-002] when ρ = 0.95 and ρ = 0.99 respectively.

Comparison between RRE and RRRE

When ρ = 0.95 and 0.99, the restricted ridge regression parameter *k* was found as 0.093 and 0.295 respectively, using the methods suggested in section 3.1 (3). Using these *k* values, the RRRE was obtained.

The mse parameter vector of RRE and RRRE

The scalar mean square error of the parameter vector **(mse)** is calculated using the expressions given in section 3 (iii) & 3 (iv) and it is given below Table 3.

Table 3: The scalar mean square error of RRE and RRRE

	ρ = **0.95**		ρ = **0.99**	
	RRE	**RRRE**	**RRE**	**RRRE**
mse	0.03857	0.02605	0.01901	0.01883

RRRE has smaller mse than the RRE at both situations ρ = 0.95 and 0.99.

The MSE parameter vector of RRE and RRRE

When the biased RRRE is preferable to the biased RRE, the matrix MSE(RRE) – MSE(RRRE) must be non-negative definite matrix. So we will consider the eigenvalues of MSE(RRE)-MSE(RRRE).

The eigenvalues of the matrix MSE(RRE) – MSE(RRRE) are:
[-9.9355e-005, 4.8944e-005, 4.3287e-003, 7.7787e-003, 1.0298e-002] and [-9.9809e-005, 4.8030e-004, 3.1433e-003, 3.2162e-003, 3.3266e-003] when $\rho = 0.95$ and $\rho = 0.99$ respectively.

The above eigenvalues are positive but two eigenvalues (-9.9355e-005, 4.8944e-005) are closed to zero when $\rho = 0.95$, and one eigenvalue (-9.9809e-005) is closed zero when $\rho = 0.99$. So it can be seen that matrix MSE(RRE) – MSE(RRRE) is a non negative definite matrix.

The mse parameter vector of RLSE and RRRE

The scalar mean square error of the parameter vector **(mse)** is calculated using the expressions given in section 3 (ii) & 3 (iv) and it is given below Table 4.

The MSE parameter vector of RLSE and RRRE

If biased RRRE is better than the unbiased RLSE then MSE(RLSE) – MSE(RRRE) is non-negative definite matrix. This implies that the eigenvalues of MSE(OLSE) – MSE(RRE) is greater than or equal to zero.

The eigenvalues of the matrix MSE(RLSE) – MSE(RRRE) are:
[-4.7994e-005, -8.3333e-006, 6.3524e-005, 1.4101e-003, 5.1119e-003] and [-5.5169e-004, 3.0820e-009, 3.5960e-004, 9.0133e-004, 1.5019e-003] when $\rho = 0.95$ and $\rho = 0.99$ respectively.

Table 4: The scalar mean square error of RLSE and RRRE

	$\rho = 0.95$		$\rho = 0.99$	
	RLSE	**RRRE**	**RLSE**	**RRRE**
mse	0.03581	0.02929	0.02594	0.02373

When $\rho = 0.95$ and 0.99 the *mse* of RRRE has smaller than the RLSE.

From the **mse** and **MSE** results it was found that the biased RRRE is superior over the biased RRE when $0<k<0.00704$, and $0<k<0.00153$ for $\rho = 0.95$ and 0.99 respectively.

Comparison between RLSE and RRRE

When $\rho = 0.95$ and 0.99, the restricted ridge regression parameter k was found as 0.0068 and 0.0014 respectively, using the methods suggested in section 3.1 (4). Using these k values, the RRRE was obtained.

If we consider the above eigenvalues three values are closed to zero (-4.7994e-005, -8.3333e-006, 6.3524e-005) and two values are positive when $\rho = 0.95$, and two eigenvalues (-5.5169e-004, 3.0820e-009) are closed zero and three value are positive when $\rho = 0.99$. So it can be seen that matrix MSE(RLSE) – MSE(RRRE) is a non negative definite matrix.

According to the **mse** and **MSE** results it was found that the biased RRRE is superior over the unbiased RLSE when $0<k<0.00704$, and $0<k<0.00153$ for $\rho = 0.95$ and 0.99 respectively.

Comparison between MRRE and RRRE

When the restrictions are true at two different levels of multicollinearity ρ = 0.95 and 0.99, the restricted ridge regression and modified ridge regression same biasing parameter k was found as 0.01 and 0.00028 respectively, using the methods suggested in section 3.1 (5). Using these k values, the MRRE and RRRE were obtained.

The mse parameter vector of MRRE and RRRE

The scalar mean square error of the parameter vector (mse) is calculated using the expressions given in section 3 (iv) & 3 (v) and it is given below Table 5.

From the above eigenvalues, three values (-9.4351e-004, -2.1975e-009, 5.9518e-008) are closed to zero and two eigenvalues positive when ρ = 0.95, similarly three eigenvalues (-9.0567e-008, 3.0215e-008, 6.8955e-008) are closed zero and two value are positive when ρ = 0.99. So it can be seen that matrix MSE(MRRE) – MSE(RRRE) is a non negative definite matrix.

From the above **msc** and **MSE** results it was found that the biased RRRE is superior over the biased MRRE when 0<k<0.03673, and 0<k<0.02.8983X10^{-4} for ρ = 0.95 and 0.99 respectively.

Table 5: The scalar mean square error of MRRE and RRRE

	ρ = 0.95		ρ = 0.99	
	MRRE	**RRRE**	**MRRE**	**RRRE**
msc	0.04443	0.02713	0.05204	0.02884

RRRE has smaller mse than the MRRE at the same biasing parameter when = 0.95 and 0.99.

The MSE parameter vector of MRRE and RRRE

The biased RRRE is preferable to the biased MRRE, the matrix MSE(MRRE) – MSE(RRRE) must be non-negative definite matrix. So we have to consider the eigenvalues of the matrix MSE(MRRE) – MSE(RRRE).

The eigenvalues of the matrix MSE(MRRE) – MSE(RRRE) are: [-9.4351e-004, -2.1975e-009, 5.9518e-008, 8.4002e-003, 9.8423e-003] and [-9.0567e-008, 3.0215e-008, 6.8955e-008, 8.7652e-003, 1.6765e-002] when ρ = 0.95 and ρ = 0.99 respectively.

Conclusion

When the independent variables are correlated among themselves, multicollinearity exists and the restrictions are indeed correct then selection of the above estimators are based on different conditions, and this study reveals the way to handle the theoretical results in a practical situation.

The results also demonstrate the suitability of theoretical results when comparing the scalar mean square error of the parameter vector (**mse**) and matrix mean square error of the parameter vector (**MSE**) of the derived estimators.

When comparing the estimator at two different levels of multicillinearity (ρ = **0.95** and **0.99**) and the restrictions indeed correct the following results were obtained.

01. The unbiased RLSE is better than the unbiased OLSE.

02. The biased RRE is better than the unbiased OLSE when the biasing parameter k ranges between $0<k<0.02832$ and $0<k<0.02354$ when $\rho = 0.95$ and 0.99 respectively.

03. The biased RRRE is better than the biased RRE when the biasing parameter k ranges between $0\leq k\leq 0.093$ and $0\leq k\leq 0.295$ when $\rho = 0.95$ and 0.99 respectively.

04. The unbiased RRRE is better than the unbiased RLSE when the biasing parameter k ranges between $0<k<0.0074$ and $0<k<0.00153$ when $\rho = 0.95$ and 0.99 respectively.

05. The biased RRRE is better than the biased MRRE when the biasing parameter k ranges between $0<k<0.03673$ and $0<k<2.8983X10^{-4}$ when $\rho = 0.95$ and 0.99 respectively.

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