INTERPOLATION: A TAYLOR POLYNOMIAL APPROACH

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Abstract:

Interpolation is a technique that calculates the unknown values from known given values with in the certain range. Whereas the process of calculating unknown values beyond the certain given range is called extrapolation. However, the term interpolation includes extrapolation. Many operators require weak interpolation theory to accurately describe their cartographic properties. For example, Anthony P.Austin (2016) discussed several topics related to interpolation and how it is used in numerical analysis. Biswajit Das, Dhritikesh Chakrabarty (2016) had developed an interpolation formula derived from Lagrange's interpolation formula. The formula obtained had been applied to represent the numerical data, on the total population of India by a suitable polynomial. Slawomir Sujecki (2013) proposed an extension of the concept of Taylor series to arbitrary function that are physically meaningful. Prime objective of our work is to construct a model for interpolation to get better approximation compare to some existing method. In this research we aimed to approximate the Taylor polynomial of unknown function by known data set. We obtained system of equations by substituting known data set to the Taylor polynomial and found the derivatives needed for interpolation model using the system of equations. Then we intended to compare the proposal method with some existing method taking single polynomial, trigonometric and exponential functions as test functions. Also, we compared our model with well-known existing method, polynomial interpolation. The proposed model overlaps on the polynomial function and exponential function when 5 points are taken with even comparatively larger step size. The error of our model is less than the error of polynomial interpolation throughout the range in both cases. However, proposed model deviates from test function when using small number of feed data for trigonometric function. When increased the feed data, our model and the trigonometric function fall on the same curve whereas the usual polynomial interpolation deviates much from the trigonometric function. Consequently, we can conclude that proposed model performs better than polynomial interpolation for polynomial, trigonometric and exponential functions. And in the case of increased amount of feed data, we record improved accuracy at interpolating procedure.

Keywords: Interpolation, Taylor series, Representation of numerical data.

1. Introduction

Interpolation is a technique that calculates the unknown values from known given values with in the certain range. Whereas the process of calculating unknown values beyond the certain given range is called extrapolation. However, the term interpolation includes extrapolation. Extrapolating to a range that is too large can be dangerous if it is not guaranteed that the relationship between the variables is maintained throughout the range.

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The use of interpolation theory in various field of analysis is well known. Many operators require weak interpolation theory to accurately describe their cartographic properties. For example, Anthony P.Austin (2016) discussed several topics related to interpolation and how it is used in numerical analysis. Biswajit Das, Dhritikesh Chakrabarty (2016) had developed an interpolation formula derived from Lagrange's interpolation formula. The formula obtained had been applied to represent the numerical data, on the total population of India by a suitable polynomial. Slawomir Sujecki (2013) proposed an extension of the concept of Taylor series to arbitrary function that are physically meaningful.

Prime objective of our work is to construct a model for interpolation to get better approximation compare to some existing method. This model aims to approximate the Taylor polynomial of unknown function by known data set. Then we intend to compare the proposed method with some existing method by taking a single polynomial, a trigonometric and an exponential function as test functions.

2. Methodology

Suppose that there are $n+1$ number of points with step size h in domain X and $x_m = x_0 + mh$ for $m = 0,1,2 \cdots n$. Let $y_0, y_1, \cdots\ y_m\ \cdots y_n$ are corresponding values in the co-domain Y satisfying some unknown function $y = f(x)$ and $\{x_m, f(x_m)\}_{m=0}^{n}$. General formula of our interpolation model by Taylor expansion around left point of the given data set is given by

$$f_{model}(x) = \sum_{k=0}^{n} \frac{(x - x_0)}{k!} y_0^{(k)} \tag{1}$$

for any $m = 0\ ,1\ ,2\ ,\ \cdots\ n.$

This has the form

$$y_m = f(x_m) = y_0^{(0)} + (mh).y_0^{(1)} + \frac{(mh)^2}{2!} y_0^{(2)} + \frac{(mh)^3}{3!} y_0^{(3)} + \cdots + \frac{(mh)^n}{n!} y_0^{(n)}.$$

We have to generate $y_0^{(0)}, y_0^{(1)}, y_0^{(2)}, \cdots, y_0^{(m)}, \cdots, y_0^{(n)}$ to construct Taylor expansion model for interpolation. In order to do that, we solve this system of $n+1$ equations by expressing in matrix form.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & h & \frac{(h)^2}{2!} & \cdots & \frac{(h)^{n-1}}{(n-1)!} & \frac{(h)^n}{n!} \\ 1 & 2h & \frac{(2h)^2}{2!} & \cdots & \frac{(2h)^{n-1}}{(n-1)!} & \frac{(2h)^n}{n!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & (n-1)h & \frac{[(n-1)h]^2}{2!} & \cdots & \frac{[(n-1)h]^{n-1}}{(n-1)!} & \frac{[(n-1)h]^n}{n!} \\ 1 & nh & \frac{(nh)^2}{2!} & \cdots & \frac{(nh)^{n-1}}{(n-1)!} & \frac{(nh)^n}{n!} \end{bmatrix} \begin{bmatrix} y_0^{(0)} \\ y_0^{(1)} \\ y_0^{(2)} \\ \vdots \\ y_0^{(n-1)} \\ y_0^{(n)} \end{bmatrix}$$

This can be written in the form

$$Y_{(n+1)\times 1} = A_{(n+1)\times(n+1)}B_{(n+1)\times 1} \tag{2}$$

Multiplying (2) both sides by $A^{-1}_{(n+1)\times(n+1)}$, we get

$$B_{(n+1)\times 1} = A^{-1}_{(n+1)\times(n+1)}Y_{(n+1)\times 1}$$

Finally, we are able to generate the general formula for interpolation using the matrix elements of $B_{(n+1)\times 1}$ and (1)

$$f_{model}(x) = \sum_{k=0}^{n} \frac{(x-x_0)}{k!} B_0^{(k)} \tag{3}$$

3. Results and Discussion

3.1 Experiment with polynomial function

First, we take a polynomial function as the test function. For this, we consider $y = f(x) = 2x^3 + x^2 + 2x + 1$.We randomly choose a range [-4, -3] as domain X which generate 5 points with uniform step size $h = 0.25$ .We found the matrix

$$\boldsymbol{B} = [-119 \quad 90.003 \quad -46.016 \quad 12.032 \quad 0]^{\boldsymbol{T}}$$

to construct our model. The final form of our proposed interpolation model is

$$\begin{aligned} f_{model}(x) &= -119 + 90.003(x+4) - \frac{46.016}{2!}(x+4)^2 + \frac{12.032}{3!}(x+4)^3 \\ &= -119 + 90.0003(x+4) - 23.008(x+4)^2 + 3.008(x+4)^3. \end{aligned}$$

For visual understanding, we plotted this in figure 1. It is observed that our proposed model overlaps on the test function within the range

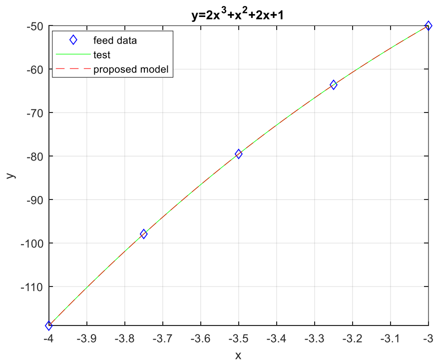


Figure 1: test function and proposed model using 5 number of feed data

3.2 Experiment with trigonometric function

Here,we choose the test function as the trigonometric function $y = f(x) = \sin x$ in the range $[-\pi, \pi]$ with 5 equidistant points with step size $h = \frac{\pi}{2}$. In this case, the matrix $\boldsymbol{B}$ is

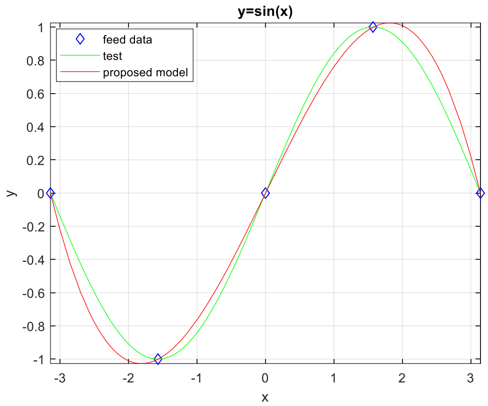
$$\boldsymbol{B} = [0 \quad -1.6917 \quad 1.6211 \quad -0.516 \quad 0]^{\boldsymbol{T}}$$

and the final model is

$$f_{model}(x) = -1.6917(x+\pi) + 0.8106(x+\pi)^2 - 0.086(x+\pi)^3.$$

This is illustrated in figure 2 (a) and observed that the proposed model has a significant variation with the graph of actual sin function. Hence, we increased the number of points to

9 and did same procedure. In figure 2 (b), we then observed that the test function coincides with the test function.



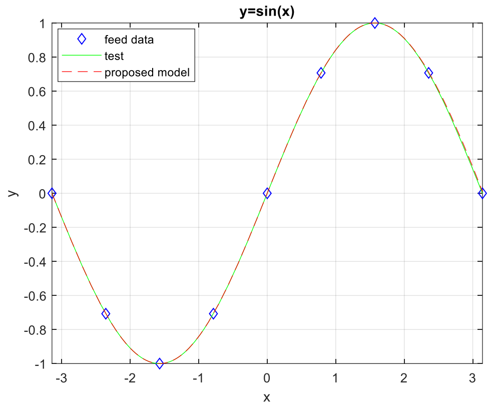


Figure 2(a) Graph of test function and proposed model using feed points

Figure 2(a) Graph of test function and proposed model using 9 feed points

3.3 Experiment with exponential function.

Finally, we considered an exponential function $y = f(x) = e^x$ in the range [-1, 0] with uniform step size $h = 0.25$. The matrix B and the model were found as

$$\boldsymbol{B} = [0.3699 \quad 0.3674 \quad 0.3751 \quad 0.3096 \quad 0.6129]^T$$

and

$$f_{model}(x) = 0.3699 + 0.3674(x+1) + 0.1876(x+1)^2 + 0.0516(x+1)^3 + 0.0255(x+1)^4$$

This is illustrated in figure 3.

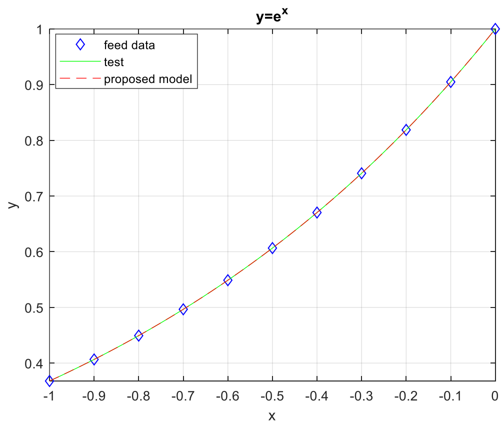


Figure 3. Graph of test function and proposed model

4. Conclusion

The proposed model overlaps on the polynomial function and exponential function when 5 points are taken with even comparatively larger step size. The error of our model is less than the error of polynomial interpolation throughout the range in both cases. However, the proposed model deviates from test function when using small number of feed data for trigonometric function. When increased the feed data, our model and the trigonometric function fall on the same curve whereas the usual polynomial interpolation deviates much from the trigonometric function.

Consequently, we can conclude that proposed model performs better than polynomial interpolation for polynomial, trigonometric and exponential functions. And in the case of increased amount of feed data, we record improved accuracy at interpolating procedure.

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