Fourier method for one dimensional parabolic inverse problem with Neumann boundary condition

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Introduction

Heat conduction problems have achieved substantial popularity in science and industrial fields and play a very important role, such as resources exploration, aerospace engineering, atmosphere measure, ocean engineering, quantum mechanics, etc. Industry-related heat conduction research has a great significance and is mostly regarded to the inverse problem of heat conduction equation such as by determining boundary, initial or the internal data of the medium of heat transfer, a known temperature at the internal or the boundary point of the domain can be controlled. Usually solving the inverse problem of the heat conduction equation is ill-conditioned and the small perturbation of data will lead to a problem of huge error.

There are several methods available in the literature to solve inverse heat conduction problems. Numerical methods such as the finite element method [1], finite volume method [2], and difference schemes[3] have been used by many researchers. Approximations using the moving least square method, variational iteration method, cubic-B spline method, and polynomial regression model are some other methods used to solve inverse problems. But there is less research on solving inverse parabolic problems using the Fourier method where [4] presents the Fourier series analysis of the inverse problem of finding the coefficient of the lowest term within the heat equation with a non-local Wentzell–Neumann boundary and integral over determination conditions. The novelty of our work is, we use the Fourier method to construct the control parameters, initial condition, and the source term.

In this study, we consider the following uncontrolled heat equation with homogeneous Neumann boundary conditions in a finite-dimensional interval:

$$u_t(x,t)-\propto u_{xx}(x,t)=0 \quad 0\leq x\leq L, t\geq 0, \text{-(1)}$$
$$u_x(0,t)=u_x(L,t)=0 \quad t\geq 0,$$
$$u(x,0)=f(x) \quad 0\leq x\leq L,$$

and find the control parameters, initial temperature g(x) and the heat source $\Phi(x,t)$ such that the point evaluation $u(x_0,t)$ tracks the desired signal $F(t)\in C(0,T)$, where the controlled system is given by,

$$u_t(x,t)-\propto u_{xx}(x,t)=\Phi(x,t) \quad 0\leq x\leq L, t\geq 0,$$
$$u_x(0,t)=u_x(L,t)=0 \quad t\geq 0,$$
$$u(x,0)=g(x) \quad 0\leq x\leq L,$$
$$u(x_0,t)=F(t).$$

Here F(t) is a known function.

Objectives of this study are,

- to construct the control parameters, the initial condition, and source term so that the point evaluation at an internal point in the domain will track a known function.
- to validate our findings using COMSOL simulations.

Methodology

To find the control parameters of the heat equation with Neumann boundary conditions the uncontrolled heat equation with homogeneous Neumann boundary conditions was solved using the method of separation of variables,

$$u(x,t)=\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n^2\pi^2\alpha t}{L^2}\right)}$$

By considering the controlled initial boundary value problem (1), let the Fourier expansions of internal temperature, initial temperature and heat source be

$$u(x,t)=\sum_{n=1}^{\infty} a_n(t)\cos\left(\frac{n\pi x}{L}\right),\text{-(2)}$$

$$g(x)=\sum_{n=1}^{\infty} b_n\cos\left(\frac{n\pi x}{L}\right),\text{-(3)}$$

$$\Phi(x,t)=\sum_{n=1}^{\infty} c_n(t)\cos\left(\frac{n\pi x}{L}\right).\text{-(4)}$$

Differentiating (2) with respect to t and twice with respect to x, substituting the results in (1) along with (4) and by using the orthogonality of $\cos\left(\frac{n\pi x}{L}\right)$, a first-order ODE can be obtained as follows:

$$a_n'(t)+\alpha\left(\frac{n\pi}{L}\right)^2 a_n(t)-c_n(t)=0\text{-(5)}.$$

At the fixed point $x=x_0$, $u(x,t)=u(x_0,t)$. Then (2) takes the form,

$$u(x_0,t)=\sum_{n=1}^{\infty} a_n(t)\cos\left(\frac{n\pi x_o}{L}\right)$$

Hence the Fourier coefficient takes the form,

$$a_n(t)=\frac{2}{L}\int_0^L u(x_0,t)\cos\left(\frac{n\pi x_0}{L}\right)dx.\text{-(6)}$$

Substituting $u(x_0,t)=F(t)$ in (6) and by integrating we obtain the Fourier coefficient of internal temperature,

$$a_n(t)=2F(t)\cos\left(\frac{n\pi x_0}{L}\right).\text{-(7)}$$

Substituting $a_n(t)$ and $a_n'(t)$ in (5), the Fourier coefficient of heat source be,

$$c_n(t)=2F'(t)\cos\left(\frac{n\pi x_0}{L}\right)+\alpha\left(\frac{n\pi}{L}\right)^2 2F(t)\cos\left(\frac{n\pi x_0}{L}\right).\text{-(8)}$$

From (1) and (8), heat source can be obtained as follows,

$$\Phi(x,t)=\sum_{n=1}^{\infty}\left[2F'(t)\cos\left(\frac{n\pi x_0}{L}\right)+\alpha\left(\frac{n\pi}{L}\right)^2 2F(t)\cos\left(\frac{n\pi x_0}{L}\right)\right]\cos\left(\frac{n\pi x}{L}\right).$$

Comparing coefficients of (2) and (3) at t=0, and by (7), we obtain the Fourier coefficient of initial temperature,

$$b_n=a_n(0)=2F(0)\cos\left(\frac{n\pi x_0}{L}\right).\text{-(9)}$$

Hence from (2) and (9), the initial temperature can be obtained as follows,

$$g(x)=2F(0)\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x_0}{L}\right)\cos\left(\frac{n\pi x}{L}\right).$$

Numerical simulation;

From the above results, the controlled system (1) can be modified to,

$$u_t(x,t)-\propto u_{xx}(x,t)=\sum_{n=1}^{\infty}\left[2F'(t)\cos\left(\frac{n\pi x_0}{L}\right)+\alpha\left(\frac{n\pi}{L}\right)^2 2F(t)\cos\left(\frac{n\pi x_0}{L}\right)\right]\cos\left(\frac{n\pi x}{L}\right),\text{-(10)}$$

$$u_x(0,t)=u_x(L,t)=0,$$

$$u(x,0)=2F(0)\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x_0}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\quad 0\le x\le L, t\ge 0.$$

Choose a domain [0,l] in COMSOL geometry. Solve (10) and plot $u(x_0,t)$, and F(t) on the same figure.

i) Track $F(t)=\sin(t)+\cos(t)$ at $x_0=0.75$ to validate the trigonometric functions satisfy the results
ii) Track $F(t)=2t$ at $x_0=0.75$ to validate the linear functions satisfy the results

Results and Discussion

Fourier coefficients of the approximations of the initial condition and source term were determined by derived ODEs as given below:

Initial temperature

$$g(x)=2F(0)\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x_0}{L}\right)\cos\left(\frac{n\pi x}{L}\right),$$

Heat source

$$\Phi(x,t)=\sum_{n=1}^{\infty}\left[2F'(t)\cos\left(\frac{n\pi x_0}{L}\right)+2\alpha\left(\frac{n\pi}{L}\right)^2\cos\left(\frac{n\pi x_0}{L}\right)F(t)\right]\cos\left(\frac{n\pi x}{L}\right).$$

To validate the results, we simulate in COMSOL. In there, we solve the controlled heat equation with the above results and track the temperature values at an interior point in the geometry for a given known signal.

Tracking $F(t)=\sin(t)+\cos(t)$ at $x_0=0.75$;

We solve the following heat problem,

$$u_t(x,t)-u_{xx}(x,t)=[2(\cos(t)-\sin(t))\cos(0.75\pi)+2\pi^2(\sin(t)+\cos(t))\cos(0.75\pi)]\cos(\pi x),$$
$$u_x(0,t)=u_x(1,t)=0,$$
$$g(x)=2\cos(0.75\pi)\cos(\pi x),$$

in COMSOL for $0\le x\le 1$, $0\le t\le 1$, when $n=1$, $\alpha=1$ and $L=1$.

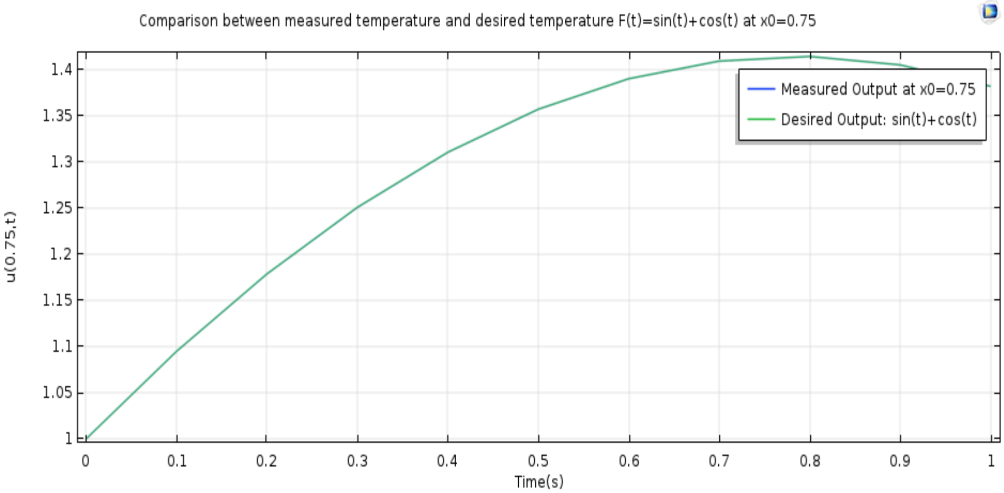

Figure 1. The comparison between the measured temperature and desired temperature at $x_0=0.75$ for $F(t)=\sin(t)+\cos(t)$.

The above figure shows that the measured temperature and the desired temperature F(t) overlap which validates the results for any polynomial as polynomials can be represented using sine and cosine series.

Tracking $F(t)=2t$ at $x_0=0.75$;

We solve the following heat problem,

$$u_t(x,t)-u_{xx}(x,t)=[4\cos(0.75\pi]+4\pi^2 t\cos(0.75\pi)]\cos(\pi x),$$
$$u_x(0,t)=u_x(1,t)=0,$$
$$g(x)=0,$$

in COMSOL for $0\le x\le 1$, $0\le t\le 1$, when $n=1$, $\alpha=1$ and $L=1$.

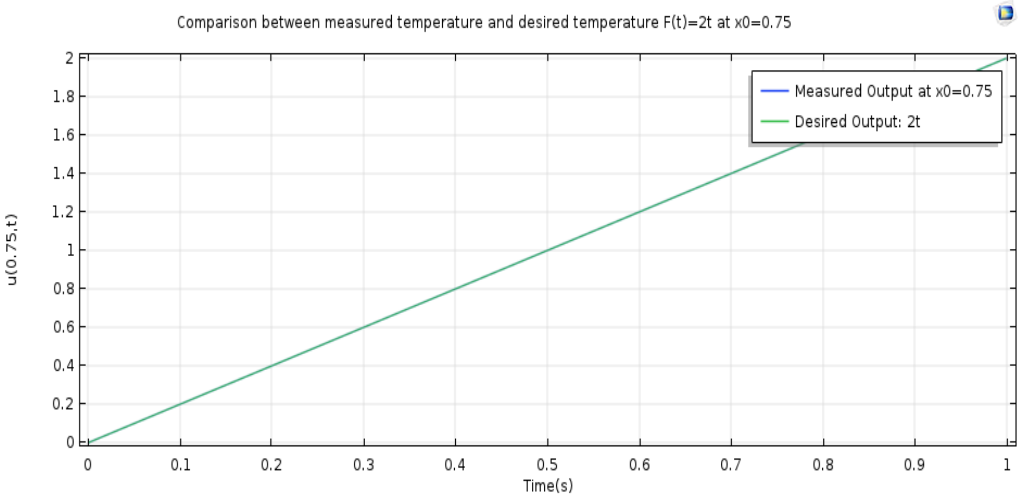

Figure 2. The comparison between the measured temperature and desired temperature at x_0=0.75 for F(t)=2t.

The above figure shows that the measured temperature and the desired temperature F(t) overlap which validates the results for any linear function. The trigonometric, linear, and nonlinear functions also give an overlapped plot, as shown above, proving that the results are valid for any known linear, trigonometric, and polynomial signals. The control parameters obtained can be used to track any known point which validates the results for any given point in the domain considered. Here we did our study only for n=1 and we will further study the case n>1 in the future. Even though we used homogeneous Neumann boundary conditions, the results are applicable for non-homogeneous Neumann boundary conditions, as we can transform the problem to a homogeneous Neumann boundary value problem.

Conclusion

In this work, the one-dimensional inverse parabolic problem with Neumann boundary conditions has been solved using the Fourier method. The control parameters, the heat source, and the initial condition have been obtained such that the point evaluation $u(x_0,t)$ tracks the desired signal $F(t)\in C(0,T)$ in the controlled system. The simulation results show that the control parameters we found can be used to track any given point in the domain considered for n=1. Finally, it can be concluded that the Fourier method is an effective method to solve inverse parabolic problems.

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