Exact solutions of Einstein - Maxwell field equations describing relativistic anisotropic stellar models

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Introduction

Exact solutions of the Einstein-Maxwell field equations are of crucial importance in modeling highly compact stars in general relativity. The models generated are used to describe relativistic spheres with strong gravitational fields as is the case in neutron stars. It is for this reason that many investigators use a variety of techniques to attain exact solution. A comprehensive list of Einstein-Maxwell solutions, satisfying a variety of criteria for physical admissibility is provided by Ivanov [3]. Mafa Takisa and Maharaj [4] utilized a linear equation of state to come up with regular solutions of anisotropic spherically symmetric charged distributions. Komathiraj and Sharma [5, 6] presented a general class of Einstein-Maxwell solutions in the presence of anisotropic stress. Our intention is to achieve simple forms for the solutions that are physically reasonable and model a charged anisotropic relativistic sphere. The main objective of this work is two-fold. Firstly, we seek to model a charged relativistic sphere with anisotropic matter which is physically acceptable. We require that the gravitational, electromagnetic and matter variables are finite, continuous and well behaved in the stellar interior, the interior metric should match smoothly with the exterior Reissner-Nordstrom metric, the speed of sound is less than the speed of light, and the solution is stable with respect to radial perturbations. Secondly, we seek to regain an isotropic solution of Einstein equations which satisfy the relevant physical criteria when the anisotropic factor vanishes. A new class of Einstein-Maxwell solutions are found that contain familiar models which can be regained for different choices of the metric function and the electric field or anisotropic factor.

Methodology

In standard coordinates $(x^a) = (t, r, \theta, \phi)$, the line element of a spherically symmetric space-time is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (1)$$

where λ and ν are function of the radial coordinate r.

Field Equations

The Einstein- Maxwell field equations for charged anisotropic fluids $(E \neq 0,\ p_r \neq p_t)$ can be written for the line element (1) as:

$$\frac{1}{r^2}\left(1 - e^{-2\lambda}\right) + \frac{2\lambda'}{r}e^{-2\lambda} = \rho + \frac{1}{2}E^2, \qquad (2.1)$$

$$-\frac{1}{r^2}\left(1 - e^{-2\lambda}\right) + \frac{2\nu'}{r}e^{-2\lambda} = p_r - \frac{1}{2}E^2, \qquad (2.2)$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p_t + \frac{1}{2}E^2, \qquad (2.3)$$

$$p_r + \Delta = p_{t,} \tag{2.4}$$

where ρ is the density, p_r is the radial pressure, p_t is the tangential pressure, E is the electric field and Δ is the anisotropic factor and primes denote the differentiation with respect to the radial coordinate r. The system (2) can be written as an equivalent form by introducing the transformation introduced by Durgapal and Bannerji [7]:

$$x = Cr^2 \ , Z(x) = e^{-2\lambda(r)} \ , \ A^2y^2(x) = e^{2\nu(r)}, \tag{3}$$

where Z, y are new metric function and C is a real constant. With the help of the transformation (3), the system (2) can be replaced as

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C}, \tag{4.1}$$

$$\frac{4Z\dot{y}}{y} + \frac{Z-1}{x} = \frac{p_r}{C}, \tag{4.2}$$

$$4Zx^2\ddot{y} + 2x^2\dot{Z}\dot{y} + \left(\dot{Z}x - Z + 1 - \frac{E^2x}{C} - \frac{\Delta x}{C}\right)y = 0, \tag{4.3}$$

$$p_r + \Delta = p_{t,} \tag{4.4}$$

where dots denote the differentiation with respect to the variable x. The system (4) governs the gravitational behavior of the charged relativistic sphere with anisotropic stress. The system (4) has four equation with seven unknowns. Hence, we have the freedom to choose specific values for any three of the variables.

Integration Procedure

We solve the Einstein-Maxwell system (4) by making explicit choices for Z, Δ and E.

$$Z(x) = \frac{1}{1+ax}, \quad \Delta(x) = \frac{a^2C\beta x}{(1+ax)^2}, \qquad E(x) = \frac{a^2C(\alpha-\beta)x}{(1+ax)^2}, \tag{5}$$

where a, α and β are real parameters. The choices are well behaved and physically reasonable. Upon substituting the choices (5) in equation (4.3) we obtain

$$4(1+ax)\ddot{y} - 2a\dot{y} + a^2(1-\alpha)y = 0 \tag{6}$$

which is the second order linear differential equation. Two categories of solution are possible for $1-\alpha = 0$ and $1 - \alpha \neq 0$

Special case: elementary functions $(1-\alpha = 0)$

In this case, the equation (6) can be separable and we obtain the solution

$$y(x) = c_1\frac{(2+2ax)^{\frac{3}{2}}}{3a} + c_2, \tag{7}$$

where c_1 and c_2 are arbitrary constants of integration. Hence the complete solution of the Einstein-Maxwell system (4) is then given by

$$e^{2\nu} = A^2\left[\frac{c_1(2+2ax)^{\frac{3}{2}}}{3a} + c_2\right]^2 \tag{8.1}$$

$$e^{2\lambda} = 1 + ax \tag{8.2}$$

$$\rho = \frac{aC(3+ax)}{(1+ax)^2} \tag{8.3}$$

$$p_r = \left[\frac{aC\left(2(-5+ax)\sqrt{2+2ax}c_1+3ac_2\right)}{(1+ax)\left(2\sqrt{2}(1+ax)^{\frac{3}{2}}c_1+3ac_2\right)}\right] \quad (8.4)$$

$$p_t = \left[\frac{a^2Cx\beta}{(1+ax)^2} - \frac{aC\left(2(-5+ax)\sqrt{2+2ax}c_1+3ac_2\right)}{(1+ax)\left(2\sqrt{2}(1+ax)^{\frac{3}{2}}c_1+3ac_2\right)}\right] \quad (8.5)$$

The solution (8) is written completely in terms of elementary functions and easy to do the physical analysis.

General case: Bessel functions

With, $1-\alpha \neq 0$ equation (6) is difficult to solve. Consequently, we introduce the transformation

$$X = 1 + ax \quad (9.1)$$

$$y(x) = Y(X) \quad (9.2)$$

With the help of (9), the differential equation (6) becomes

$$4\mathrm{X}\frac{\mathrm{d}^2\mathrm{Y}}{dX^2} - 2\frac{dY}{dX} + (1-\alpha)Y = 0 \quad (10)$$

We now introduce a new function $u(X)$ such that

$$Y(X) = u(X)X^m \quad (11)$$

where m is a constant. Substitution of (11) in equation (10) gives

$$4\mathrm{X}^2\frac{d^2u}{dX^2} + \mathrm{X}(8m-2)\frac{du}{dX} + [4m^2 - 6m + (1-\alpha)X]u \quad (12)$$

It is convenient at this point to introduce the new variable w as follows:

$$w = X^\gamma \quad (13)$$

Upon substituting w in (12), we obtain

$$4w^2\gamma^2\frac{d^2u}{dw^2} + 2w\gamma(2\gamma+4m-3)\frac{du}{dw} + \left[4m^2 - 6m + (1-\alpha)w^{\frac{1}{\gamma}}\right]u = 0 \quad (14)$$

We observe that there is considerable simplification if we make the choice $\gamma = \frac{1}{2}$ and $m = \frac{3}{4}$. Then the equation (14) becomes

$$w^2\frac{d^2u}{dw^2} + w\frac{du}{dw} + \left[-\frac{9}{4} + (1-\alpha)w^2\right]u = 0 \quad (15)$$

Consequently, we introduce the transformation

$$(1-\alpha)^{\frac{1}{2}}w = v \quad (16)$$

With this transformation the equation (15) becomes

$$v^2\frac{d^2u}{dv^2} + v\frac{du}{dv} + \left[v^2 - \left(\frac{3}{2}\right)^2\right]u = 0 \quad (17)$$

which is a Bessel differential equation of order 3/2 in terms of the new dependent variable u and independent variable v. With the assistance of Mathematica, the solution of equation (17) can be written as

$$u(v) = -\left[\frac{\sqrt{\frac{2}{\pi}}\left((vc_1+c_2)\cos v + (-c_1+vc_2)\sin v\right)}{v^{\frac{3}{2}}}\right] \quad (18)$$

The expression given in (18) in terms of the original variable x becomes

$$\mathrm{y}(x) = -\sqrt{\frac{2}{\pi}}(1+ax)^{\frac{3}{4}}\left[\frac{\left(\sqrt{(1+ax)(1-\alpha)}C_1+C_2\right)\cos\sqrt{(1+ax)(1-\alpha)} + \left(\sqrt{(1+ax)(1-\alpha)}C_2 - C_1\right)\sin\sqrt{(1+ax)(1-\alpha)}}{\left(\sqrt{(1+ax)(1-\alpha)}\right)^{\frac{3}{2}}}\right] \quad (19)$$

Hence the complete solution of the Einstein-Maxwell system (4) is then given by

$$e^{2\nu(r)} = A^2y^2, \quad e^{2\lambda(r)} = 1 + ax, \quad \rho = \frac{aC(3+ax)}{(1+ax)^2}, \qquad p_r = \frac{4C}{(1+ax)}\frac{\dot{y}}{y} - \frac{aC}{1+ax}$$

$$p_t = \frac{4C}{(1+ax)}\frac{\dot{y}}{y} - \frac{aC}{1+ax} + \frac{Ca^2\beta x}{(1+ax)^2}, \quad E^2 = \frac{Ca^2(\alpha-\beta)x}{(1+ax)^2}, \quad \Delta = \frac{Ca^2\beta x}{(1+ax)^2} \qquad (20)$$

Where y is given by (19). It is remarkable that these solutions are expressed completely as elementary functions.

Results and discussion

From our general class of solutions (20) and (8) found above it is possible to generate particular solutions found previously. Consider one example with $\beta = 0$ and $a = 1$. Then (19) becomes the model of Hansraj and Maharaj [2]. Further setting $\alpha = 0$ (19) becomes Finch and Skea [1] neutron star model. Also, it is possible to derive many models found in the past including the solution of Mafa Takisa and Maharaj [4] from our general solution (20).

Conclusion

Our purpose in this work was to find new physically reasonable exact solutions to the Einstein-Maxwell systems in the presence of anisotropic stress. We make the physical reasonable choices for one of the gravitational potentials, anisotropy factor and the electric field. We showed that the underlying equation was a Bessel equation which admits solutions in terms of elementary functions. We present a class of new solution to the Einstein-Maxwell system and many solutions found previously are in our general class of solution.

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